Math 3103 Combinatorics (Luecking) NAME:______________________
(Please print clearly)

Seventh Quiz (solutions) Due March 4, 2024

1. Answer (a)–(c) below for the following recurrence relation problem.

$$a_n - 4a_{n-1} + 4a_{n-2} = 0 \quad (n \geq 2),$$
$$a_0 = 1\,, \quad a_1 = 7\,.$$

(a) Write down the characteristic equation and its roots:

Ans: $r^2 - 4r + 4 = 0$ with double root $r = 2, 2$

(b) Write down a general solution:

Ans: $a_n = C_1 2^n + C_2 n 2^n$

(c) Find the solution that satisfies the initial conditions.

Ans: The initial conditions give $C_1 = 1$ and $2C_2 + 2C_2 = 7$ so that $C_2 = 5/2$. Thus

$$a_n = 2^n + \frac{5}{2} n 2^n$$

2. Answer (a)–(c) below for the following recurrence relation problem.

$$a_n - 4a_{n-1} + 5a_{n-2} = 0 \quad (n \geq 2),$$
$$a_0 = 2, \quad a_1 = 6\,.$$

(a) Write down the characteristic equation and its roots:

Ans: $r^2 - 4r + 5 = 0$ with roots $r = 2 \pm 1i$

(b) Write down a general solution:

Ans: Either $a_n = C_1(2+i)^n + C_2(2-i)^n$ or

$$a_n = 5^{n/2}(B_1 \cos(\theta n) + B_2 \sin(\theta n))\,, \text{ where } \theta = \cos^{-1}(2/\sqrt{5}).$$

(c) Find the solution that satisfies the initial conditions.

Ans: Complex form: the initial conditions give $C_1 + C_2 = 2$ and $(2+i)C_2 + (2-i)C_2 = 6$ so that

$C_1 - C_2 = 2/i$ giving $C_1 = 1 + 1/i$ and $C_2 = 1 - 1/i$. Thus

$$a_n = (1 + 1/i)(2+i)^n + (1 - 1/i)(2-i)^n$$

Altenatively, $B_1 = 2$ and $2B_1 + B_2 = 6$ gives $B_2 = 2$ and so

$$a_n = 5^{n/2}(2\cos(\theta n) + 2\sin(\theta n))$$