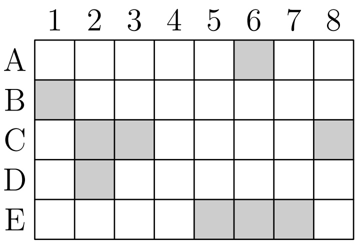
Math 3103 Combinatorics (Luecking) NAME:______________________ (Please print clearly)

Fifth Quiz (solutions) Due February 16, 2024

1. This table represents possible seating of 5 people among 8 seats. A shaded square represents a forbidden seat assignment. The rook polynomial of the shaded squares is

$$1 + 9x + 28x^2 + 36x^3 + 20x^4 + 4x^5$$

Using this information, how many allowable ways are there to seat all five people? Your final answer should be in elementary form.



Ans: $\frac{8!}{3!} - 9\frac{7!}{3!} + 28\frac{6!}{3!} - 36\frac{5!}{3!} + 20\frac{4!}{3!} - 4\frac{3!}{3!}$

2. Consider the following equation. The variables w_1, w_2, and w_3 are integers that must satisfy the given conditions.

$$w_1 + w_2 + w_3 = n$$

$$10 \le w_1 \longrightarrow x^{10} + x^{11} + \cdots = \frac{x^{10}}{1-x}$$

$$0 \le w_2 \le 19 \longrightarrow 1 + x + x^2 + \cdots + x^{19} = \frac{1-x^{20}}{1-x}$$

$$10 \le w_3 \le 29 \longrightarrow x^{10} + x^{11} + \cdots + x^{29} = \frac{x^{10} - x^{30}}{1-x}$$

(a) Write out the generating function for the number of solutions of this equation. Write your answer as a quotient in which the denominator is a power of $(1-x)$ and the numerator is a polynomial written out as a sum of different powers of x.

Ans: The generating functions for each variable are given above. The generating function for the given problem is the product of these: $F(x) = \dfrac{x^{20} - 2x^{40} + x^{60}}{(1-x)^3}$.

(b) Use the result in part (a) to find the number of solutions when $n = 80$.

Ans: $F(x) = \left(x^{20} - 2x^{40} + x^{60}\right) \sum_{j=0}^{\infty} \binom{j+2}{j} x^j$, and the terms that produce x^{80} are:

$$x^{20}\binom{62}{60}x^{60} - 2x^{40}\binom{42}{40}x^{40} + x^{60}\binom{22}{20}x^{20} = \left[\binom{62}{60} - 2\binom{42}{40} + \binom{22}{20}\right] x^{80},$$

so the answer is: $\binom{62}{60} - 2\binom{42}{40} + \binom{22}{20}$.