Math 3103 Combinatorics (Luecking) NAME:_____
(Please print clearly)

Fourth Quiz (solutions) Due February 12, 2024

Instructions: The phrase "in elementary form" means "using only numbers and the operations of addition, subtraction, multiplication, division, powers and factorials." If I *do not explicitly* request this form, then the forms $C(n,k)$, $\binom{n}{k}$, $P(n,k)$ and d_n (for derangements) are also permitted.

1. Suppose each of 7 students writes their name on a blank card, then the cards are collected and later returned randomly. Answer the following questions. Part (a) must be in elementary form (see instructions), the rest do not have to be.

   (a) In how many ways would *none of the students* get the card with their own name? Answer this part *in elementary form.*

Ans: This is a derangement of 7 objects: $d_7 = 7!\left(\frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} - \frac{1}{5!} + \frac{1}{6!} - \frac{1}{7!}\right)$

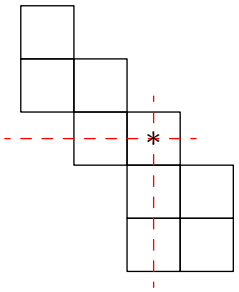
   (b) In how many ways would *at least one student* get the the card with their own name?

Ans: Number of permutations that are not derangements: $7! - d_7$

   (c) In how many ways would *exactly* 3 *of the students* get the card with their own name?

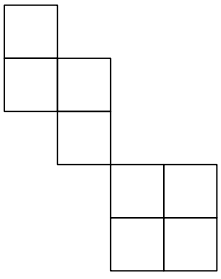
Ans: Rule of product: select which 3, then derange the remaining 4: $C(7,3) \cdot d_4$.

2. Find the rook polynomial of the chessboard below using the "remove-a-square" method and the product formula. I suggest using the square marked with a $*$. Provide your final answer as a sum of actual numbers times different powers of x.



Ans: Using the formulas, removing the square marked $*$, we get:

$$\begin{aligned} r(C_e,x) + xr(C_s,x) &= (1+4x+3x^2)(1+4x+2x^2) \\ &\quad + x(1+3x+x^2)(1+2x) \\ &= 1+9x+26x^2+27x^3+8x^4 \end{aligned}$$

C_e : 

C_s :