Math 3083 Linear Algebra (Luecking) NAME:________________________
(Please print clearly)

Twelfth Quiz (solutions) Due April 15, 2024

1. Let $\mathbf{x}_1 = \begin{pmatrix}1\\0\\0\\1\end{pmatrix}$, $\mathbf{x}_2 = \begin{pmatrix}0\\1\\1\\1\end{pmatrix}$, and $\mathbf{x}_3 = \begin{pmatrix}1\\1\\1\\0\end{pmatrix}$. Using the *standard scalar product* for the inner product on $\mathbb{R}^4$, apply the Gram-Schmidt process to find an orthonormal basis for $\mathrm{Span}(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3)$. (Don't show too much detail. Fit your answer below.)

Ans: $\mathbf{v}_1 = \mathbf{x}_1 = \begin{pmatrix}1\\0\\0\\1\end{pmatrix}$. $\mathbf{v}_2 = \mathbf{x}_2 - \mathbf{p}_1 = \begin{pmatrix}0\\1\\1\\1\end{pmatrix} - \frac{1}{2}\begin{pmatrix}1\\0\\0\\1\end{pmatrix} = \begin{pmatrix}-1/2\\1\\1\\1/2\end{pmatrix}$.

$$\mathbf{v}_3 = \mathbf{x}_3 - \mathbf{p}_2 = \begin{pmatrix}1\\1\\1\\0\end{pmatrix} - \frac{1}{2}\begin{pmatrix}1\\0\\0\\1\end{pmatrix} - \frac{3/2}{5/2}\begin{pmatrix}-1/2\\1\\1\\1/2\end{pmatrix} = \begin{pmatrix}4/5\\2/5\\2/5\\-4/5\end{pmatrix}.$$

Finally, divide each by its norm: $\mathcal{B} = \left[\begin{pmatrix}1/\sqrt{2}\\0\\0\\1/\sqrt{2}\end{pmatrix}, \begin{pmatrix}-1/\sqrt{10}\\2/\sqrt{10}\\2/\sqrt{10}\\1/\sqrt{10}\end{pmatrix}, \begin{pmatrix}2/\sqrt{10}\\1/\sqrt{10}\\1/\sqrt{10}\\-2/\sqrt{10}\end{pmatrix}\right]$

2. (a) Find the eigenvalues of the matrix $A = \begin{pmatrix}3&0&0\\0&2&1\\0&2&1\end{pmatrix}$. **Ans:** $\det(A - \lambda I) = (3-\lambda)(\lambda^2 - 3\lambda) = 0$. Solve to get $\lambda = 3, 3, 0$.

(b) For each eigenvalue of A, find a basis for its eigenspace.

Ans: For $\lambda = 3$, the matrix $A - 3I = \begin{pmatrix}0&0&0\\0&-1&1\\0&2&-2\end{pmatrix}$ reduces to $\begin{pmatrix}0&1&-1\\0&0&0\\0&0&0\end{pmatrix}$. So, the eigenspace consists of $\mathbf{x} = \begin{pmatrix}\alpha\\\beta\\\beta\end{pmatrix}$, and a basis is $\begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\1\\1\end{pmatrix}$.

For $\lambda = 0$, the matrix $A - 0I = \begin{pmatrix}3&0&0\\0&2&1\\0&2&1\end{pmatrix}$ reduces to $\begin{pmatrix}1&0&0\\0&1&1/2\\0&0&0\end{pmatrix}$. So, the eigenspace consists of all $\mathbf{x} = \begin{pmatrix}0\\-\alpha/2\\\alpha\end{pmatrix}$ and a basis is $\begin{pmatrix}0\\-1/2\\1\end{pmatrix}$.