Math 3083 Linear Algebra (Luecking) NAME:______________________________
(Please print clearly)

Tenth Quiz (solutions) Due April 3, 2024

The following problems are based in $\mathbb{R}^n$ for some values of n. Use the standard scalar product when necessary.

1. For the vectors $\mathbf{u} = \begin{pmatrix} 1 \\ 2 \\ 2 \\ 0 \end{pmatrix}$ and $\mathbf{v} = \begin{pmatrix} 1 \\ 1 \\ -1 \\ 1 \end{pmatrix}$, let $S = \operatorname{Span}(\mathbf{u}, \mathbf{v})$ and find a basis for $S^\perp$. Hint: this comes down to solving a system of equations.

Ans: We need to solve the system: $\mathbf{u}^T\mathbf{x} = 0, \quad \mathbf{v}^T\mathbf{x} = 0.$

The augmented matrix of this system is $\left(\begin{array}{cccc|c} 1 & 2 & 2 & 0 & 0 \\ 1 & 1 & -1 & 1 & 0 \end{array}\right)$.

Reduced echelon form is $\left(\begin{array}{cccc|c} 1 & 0 & -4 & 2 & 0 \\ 0 & 1 & 3 & -1 & 0 \end{array}\right)$.

This gives the equations: $x_1 = 4x_3 - 2x_4$ and $x_2 = -3x_3 + x_4$. Let $x_3 = \alpha$ and $x_4 = \beta$; the solutions are $\mathbf{x} = \begin{pmatrix} 4\alpha - 2\beta \\ -3\alpha + \beta \\ \alpha \\ \beta \end{pmatrix}$ and the basis is: $\begin{pmatrix} 4 \\ -3 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -2 \\ 1 \\ 0 \\ 1 \end{pmatrix}$.

2. Find the *least squares solution* $\hat{\mathbf{x}}$ for the *inconsistent* system:

$$\begin{aligned} x_1 - x_2 - 2x_3 &= 1 \\ x_1 - x_2 \qquad &= 3 \\ 2x_1 + x_2 + x_3 &= 7 \\ + x_2 \quad x_3 &= 5 \end{aligned}$$

Ans: Let $A = \begin{pmatrix} 1 & -1 & -2 \\ 1 & -1 & 0 \\ 2 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} 1 \\ 3 \\ 7 \\ 5 \end{pmatrix}$ so the system is $A\mathbf{x} = \mathbf{b}$, and we solve instead $A^TA\mathbf{x} = A^T\mathbf{b}$. That is, $\begin{pmatrix} 6 & 0 & 0 \\ 0 & 4 & 4 \\ 0 & 4 & 6 \end{pmatrix} \mathbf{x} = \begin{pmatrix} 18 \\ 8 \\ 10 \end{pmatrix}$.

Row reduce: $\left(\begin{array}{ccc|c} 6 & 0 & 0 & 18 \\ 0 & 4 & 4 & 8 \\ 0 & 4 & 6 & 10 \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{array}\right)$. This has solution $\hat{\mathbf{x}} = \begin{pmatrix} 3 \\ 1 \\ 1 \end{pmatrix}$.

Check this by verifying that the residual $\mathbf{b} - A\hat{\mathbf{x}} = \begin{pmatrix} 1 \\ 1 \\ -1 \\ 3 \end{pmatrix}$ is orthogonal to the columns of A.