Math 3083 Linear Algebra (Luecking) NAME:________________________
(Please print clearly)

Sixth Quiz (solutions) Due March 4, 2024

1. Let $\mathcal{E} = \left[\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right]$ be the standard ordered basis for $\mathbb{R}^2$, and let $\mathcal{B}$ be the ordered basis $\left[\begin{pmatrix} 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 3 \\ 7 \end{pmatrix} \right]$. Find the change of basis matrix (or transition matrix) from $\mathcal{E}$ to $\mathcal{B}$.

Ans: The matrix $S = \begin{pmatrix} 1 & 3 \\ 1 & 4 \end{pmatrix}$ changes basis from $\mathcal{B}$ to $\mathcal{E}$. To go the other way, we need the inverse: $S^{-1} = \begin{pmatrix} 7 & -3 \\ -2 & 1 \end{pmatrix}$.

2. Let $\mathcal{E} = \left[\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right]$ be the standard ordered basis for $\mathbb{R}^3$, and let $\mathcal{B}$ be the ordered basis $\left[\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \right]$. Find the change of basis matrix (or transition matrix) from $\mathcal{E}$ to $\mathcal{B}$.

Ans: The matrix $S = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix}$ changes basis from $\mathcal{B}$ to $\mathcal{E}$. To go the other way, we need the inverse: $S^{-1} = \begin{pmatrix} 1 & 1 & -1 \\ -1 & 0 & 1 \\ 0 & -1 & 1 \end{pmatrix}$.

Note: If we have any two ordered bases in $\mathbb{R}^3$: $\mathcal{B} = [\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3]$ and $\mathcal{C} = [\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3]$ then the problem of transition from $\mathcal{C}$ to $\mathcal{B}$ is the following: if a vector $\mathbf{v}$ satisfies

$$\mathbf{v} = \alpha_1\mathbf{v}_1 + \alpha_2\mathbf{v}_2 + \alpha_3\mathbf{v}_3 = \beta_1\mathbf{u}_1 + \beta_2\mathbf{u}_2 + \beta_3\mathbf{u}_3$$

then what matrix do I have to multiply $\begin{pmatrix} \beta_1 \\ \beta_2 \\ \beta_3 \end{pmatrix}$ by to get $\begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{pmatrix}$?

If S is the matrix made up of columns $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$ and T is the matrix made up of columns $\mathbf{u}_1$, $\mathbf{u}_2$, and $\mathbf{u}_3$, then the above equations can be written

$$\mathbf{v} = S \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{pmatrix} = T \begin{pmatrix} \beta_1 \\ \beta_2 \\ \beta_3 \end{pmatrix}$$

and so the answer is $S^{-1}T$. In the above two problems, $\mathcal{C} = \mathcal{E}$ is the standard basis and so T is the identity matrix.