Math 3103 Combinatorics (Luecking) NAME:________________________

(Please print clearly)

Eighth Quiz (solutions) Due October 20, 2023

1. For the following recurrence relation problem, find the generating function for the sequence a_n without actually finding a_n.

$a_n - 3a_{n-1} + 2a_{n-2} = 3^n\,, \quad n \geq 2\,.$
$a_0 = 2, \quad a_1 = 5.$

Ans: From the recurrence relation we get

$$\sum_{n=2}^{\infty} a_n x^n - 3\sum_{n=2}^{\infty} a_{n-1}x^n + 2\sum_{n=2}^{\infty} a_{n-2}x^n = \sum_{n=2}^{\infty}(3x)^n$$

or $F(x) - 2 - 5x - 3x(F(x) - 2) + 2x^2 F(x) = \dfrac{(3x)^2}{1-3x}$.

whence $F(x) - 3xF(x) + 2x^2 F(x) = 2 - x + \dfrac{(3x)^2}{1-3x}$.

and so $F(x) = \dfrac{2 - x + \dfrac{(3x)^2}{1-3x}}{1 - 3x + 2x^2}$.

or, simplified and factored: $F(x) = \dfrac{2 - 7x + 12x^2}{(1-3x)(1-x)(1-2x)}$.

2. Do the following computations in the ring $\mathbb{Z}_{17}$. All answers must be elements of $\mathbb{Z}_{17}$, completely simplified. Note: the operations $+$ and $\cdot$ are those defined on $\mathbb{Z}_{17}$, **not** the usual operations on integers.

(a) $10 + 8 + 4$ (b) $5 \cdot 4 \cdot 10$ (c) $(7 + 12) \cdot 8$

Ans: (a) $(10 + 8) + 4 = 1 + 4 = 5$ (b) $(5 \cdot 4) \cdot 10 = 3 \cdot 10 = 13$
(c) $(7 + 12) \cdot 8 = 2 \cdot 8 = 16$

(d) -5 (e) 9^{-1} (trial and error is OK)

Ans: (d) $-5 = 12$ (because $5 + 12 = 0$) (e) $9^{-1} = 2$, because $2 \cdot 9 = 1$