Math 3103 Combinatorics (Luecking) NAME:__________________________
(Please print clearly)

Fifth Quiz (solutions) Due October 4, 2023

1. For each of the following first-order recurrence relations, find the solution that satisfies the given initial condition.

   (a) $a_n = a_{n-1} + 7\,,\ \ n \geq 1\,,$
   $a_0 = 5\,.$

Ans: Arithmetic progression: $a_n = 5 + 7n\,.$

   (b) $a_n = 3a_{n-1}\,,\ \ n \geq 1\,,$
   $a_0 = 8\,.$

Ans: (b) Geometric progression: $a_n = 8 \cdot 3^n\,.$

   (c) $a_n = (2n+3)a_{n-1}\,,\ \ n \geq 1\,,$
   $a_0 = 4\,.$

Ans: (c) Succesive multiplications: $a_n = (2\cdot 1 + 3)\,(2\cdot 2+3)\,(2\cdot 3+3)\cdots(2n+3)\,4\,.$

   (d) $a_n = a_{n-1} + n^3\,,\ \ n \geq 1\,,$
   $a_0 = 4\,.$

Ans: (d) Succesive additions: $a_n = 4 + 1^3 + 2^3 + 3^3 + \cdots + n^3$. Or $a_n = 4 + \sum_{j=1}^{n} j^3\,.$

2. For the following second-order recurrence relation and initial conditions,

   $a_n - a_{n-1} - 2a_{n-2} = 0\,,\ \ n \geq 2\,,$
   $a_0 = 2\,,\ \ a_1 = 7\,,$

   (a) Write out the characteristic equation. **Ans:** $r^2 - r - 2 = 0$.

   (b) Find the roots of the characteristic equation. **Ans:** $r = -1$ and $r = 2$.

   (c) Write the general solution of the recurrence relation. **Ans:** $a_n = C_1(-1)^n + C_2 2^n$

   (d) Find the solution that satisfies the initial conditions.

Ans: The initial conditions become $C_1 + C_2 = 2\,,$ and $-C_1 + 2C_2 = 7\,,$ giving $C_1 = -1$ and $C_2 = 3$ so that

$$a_n = -1 \cdot (-1)^n + 3 \cdot 2^n\,.$$