Math 3103 Combinatorics (Luecking) NAME:______________________ (Please print clearly)

Final Exam A(solutions) December 13, 2023

Note: If I ask for an answer "in elementary form", that means you must write it using only numbers and the operations of addition, subtraction, multiplication, division, powers and factorials. When I **do not** explicitly request this, you may also use $C(n, k)$, $\binom{n}{k}$, $P(n, k)$ and d_n (with explicit numbers, of course).

1. A contest has 5 prizes and 17 contestants. If all 5 prizes must be awarded to some contestant, how many possible ways could this be done under each of the following conditions. All these answers must be in elementary form.

   (a) The prizes are identical, and any contestant may receive any number of them.

Ans: Combination with repetition, 5 selections 17 people: $C(5+17-1, 5) = \dfrac{21!}{5!\,(21-5)!}$

   (b) The prizes are all different, and any contestant may receive any number of them.

Ans: Select 5 times, with 17 possible choices each time: 17^5.

   (c) The prizes are all different, and no contestant may receive more than one.

Ans: Select 5 contestants and then endow each with a different prize: $P(17, 5) = \dfrac{17!}{(17-5)!}$.

   (d) The prizes are identical, and no contestant may receive more than one.

Ans: Select a subset of size 5 from a set of size 17: $C(17, 5) = \dfrac{17!}{5!(17-5)!}$.

2. Answer the following questions about the 10-letter string "ANTHRACENE", which has some repeated letters: 2 "A"s, 2 "N"s and 2 "E"s. The remaining letters occur only once each. All answers must be in elementary form.

   (a) How many different arrangements of this string are there? **Ans:** $\dfrac{10!}{2!\,2!\,2!}$

   (b) How many of the arrangements in part (a) contain **none** of the substrings "AA", "NN", "EE"?

Ans: c_1 = 'contains "AA"'; c_2 = 'contains "NN"'; c_3 = 'contains "EE"'.

$$S_1 = \binom{3}{1} N(c_1) = 3\left(\frac{9!}{2!\,2!}\right),\quad S_2 = \binom{3}{2} N(c_1c_2) = 3\left(\frac{8!}{2!}\right),\quad S_3 = N(c_1c_2c_3) = 7!, \text{ so}$$

$$N(\bar{c}_1\bar{c}_2\bar{c}_3) = N - S_1 + S_2 - S_3 = \frac{10}{2!\,2!\,2!} - 3\frac{9!}{2!\,2!} + 3\frac{8!}{2!} - 5!$$

   (c) How many arrangements of this string contain **at least one** of the three substrings "AA", "NN", "EE"?

Ans: $L_1 = S_1 - S_2 + S_3 = 3\dfrac{9!}{2!\,2!} - 3\dfrac{8!}{2!} + 7!$. (Note: *Same* S_j as in part (b))

   (d) How many arrangements of this string contain **exactly one** of the three substrings "AA", "NN", "EE"?

Ans: $E_1 = S_1 - 2S_2 + 3S_3 = 3\dfrac{9!}{2!\,2!} - 2\cdot 3\dfrac{8!}{2!} + 3\cdot 7!$. (Note: *Same* S_j as in part (b))

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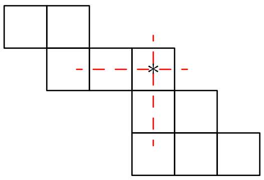
3. (a) Find the rook polynomial for the following chessboard C. Use the method that involves removing a square and the product formula. Write the answer as a sum of numbers times different powers of x, similar to the one in part (b).



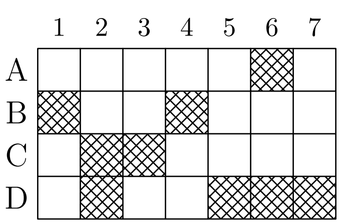
Ans: The first part of $r(C, x)$ below comes from removing the starred square, the second part from removing all the squares in the same row and column as the starred one.

$$r(C, x) = (1 + 4x + 3x^2)(1 + 5x + 4x^2) + x(1 + 2x)(1 + 3x + x^2)$$
$$= 1 + 10x + 32x^2 + 38x^3 + 14x^4 \,.$$

(b) This table represents possible seating of 4 people among 7 seats. A shaded square represents a forbidden seat assignment. The rook polynomial of the shaded squares is

$$1 + 9x + 26x^2 + 29x^3 + 10x^4$$

Using this information, how many allowable ways are there to seat all five people? This answer must be in elementary form.



Ans: $P(7, 4) - 9P(6, 3) + 26P(5, 2) - 29P(4, 1) + 10P(3, 0)$

$$= \frac{7!}{3!} - 9\left(\frac{6!}{3!}\right) + 26\left(\frac{5!}{3!}\right) - 29\left(\frac{4!}{3!}\right) + 10\left(\frac{3!}{3!}\right)$$

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4. Consider the following equation. The variables w_1, w_2 and w_3 are integers that must satisfy the given conditions.

$$w_1 + w_2 + w_3 = n$$

$$8 \le w_1 \longrightarrow x^8 + x^9 + \cdots = \frac{x^8}{1-x}$$

$$0 \le w_2 \le 19 \longrightarrow 1 + x + x^2 + \cdots + x^{19} = \frac{1-x^{20}}{1-x}$$

$$7 \le w_3 \le 26 \longrightarrow x^7 + x^2 + \cdots + x^{26} = \frac{x^7 - x^{27}}{1-x}$$

(a) Write out the generating function for the number of solutions of this equation. Write your answer as a quotient in which the denominator is a power of $(1-x)$ and the numerator is a polynomial written out as a sum of powers of x.

Ans: Take the product of the above fractions: $F(x) = \dfrac{x^{15} - 2x^{35} + x^{55}}{(1-x)^3}$.

(b) Use the result in part (a) to find the number of solutions when $n = 100$.

Ans: $F(x) = \left(x^{15} - 2x^{35} + x^{55}\right) \displaystyle\sum_{n=0}^{\infty} \binom{n+3-1}{n} x^n$, and the terms that produce x^{100} are:

$$x^{15}\binom{87}{85}x^{85} - 2x^{35}\binom{67}{65}x^{65} + x^{55}\binom{47}{45}x^{45} \text{ leading to } \binom{87}{85} - 2\binom{67}{65} + \binom{47}{45}$$

5. For the following second-order recurrence relations, find (i) the characteristic equation and its roots, (ii) the general solution, and (iii) the solution that satisfies the initial conditions.

(a) $a_n - 4a_{n-1} + 3a_{n-2} = 0\,, \quad n \ge 2\,,$
$a_0 = 0 \text{ and } a_1 = 4,.$

Ans: (i) $r^2 - 4r + 3 = 0$, roots 1 and 3. (ii) General solution: $C_1 + C_2 3^n$. (iii) The ICs, $C_1 + C_2 = 0$ and $C_1 + 3C_2 = 4$, give $C_1 = -2$ and $C_2 = 2$. Thus: $a_n = -2 + 2 \cdot 3^n$

(b) $a_n - 4a_{n-1} + 13a_{n-2} = 0\,, \quad n \ge 2\,,$
$a_0 = 0 \text{ and } a_1 = 5,.$

Ans: (i) $r^2 - 4r + 13 = 0$, roots $2 \pm 3i$. (ii) General solution: $C_1(2+3i)^n + C_2(2-3i)^n$. (iii) The ICs, $C_1 + C_2 = 0$ and $2(C_1 + C_2) + 3i(C_1 - C_2) = 5$, give $C_1 = \dfrac{5}{6i}$ and $C_2 = -\dfrac{5}{6i}$. Thus: $a_n = \dfrac{5}{6i}(2+3i)^n - \dfrac{5}{6i}(2-3i)^n$

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6. For the following nonhomogeneous recurrence relation with initial conditions, find (i) the homogeneous solution $a_n^{(h)}$, (ii) a particular solution $a_n^{(p)}$, and (iii) the solution that satisfies the initial conditions.

$$a_n - 4a_{n-1} + 4a_{n-2} = (1/9)3^n\,, \quad n \geq 2\,,$$
$$a_0 = 0 \quad \text{and} \quad a_1 = 0\,.$$

Ans: (i) The homogeneous solution is $a_n^{(h)} = C_1 2^n + C_2 n 2^n$.

(ii) A particular solution exists in the form $a_n^{(p)} = A3^n$ for some constant A. The recurrence relation gives $A - (4/3)A + (4/9)A = 1/9$, so $A = 1$ and $a_n^{(p)} = 3^n$.

(iii) The general solution is therefore $C_1 2^n + C_2 n 2^n + 3^n$. The ICs, $C_1 + 1 = 0$ and $2C_1 + 2C_2 + 3 = 0$, give $C_1 = -1$ and $C_2 = -1/2$. Thus:

$$a_n = -2^n - (1/2)n2^n + 3^n$$

7. The factorization of 2511 into primes is $2511 = 3^4(31)$ Do the following for the ring $\mathbb{Z}_{2511}$. Your final answers in parts (a) and (b) must **contain no fractions**.

(a) Find the number of units.

Ans: $\phi(2511) = 3^4(31)\left(\frac{2}{3}\right)\left(\frac{30}{31}\right) = 3^3(2)(30) = 1620$.

(b) Find the number of proper zero divisors.

Ans: $2511 - 1620 - 1 = 890$.

(c) Using the Euclidean algorithm, find $(100)^{-1}$ in this ring. Your answer must be an explicit element of $\mathbb{Z}_{2511}$.

Ans: The Euclidean algorithm gives:

$$\begin{cases} 2511 = 25(100) + 11 \\ 100 = 9(11) + 1 \end{cases} \quad \text{or} \quad \begin{cases} n = 25k \; + r_1 \\ k = 9r_1 + r_2 \end{cases}$$

where $n = 2511$, $k = 100$, $r_1 = 11$, and $r_2 = 1$. Putting $r_1 = n - 25k$ (from the top equation) into the bottom one gives
$k = 9n - 225k + r_2$ and solving for r_2: $r_2 = 226k - 9n$. If we put the values back in, this says $1 = 226 \cdot 100$ in the ring $\mathbb{Z}_{2511}$. Thus $(100)^{-1} = 226$.

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8. Let $G = u(\mathbb{Z}_{25})$, the group of units of the ring $\mathbb{Z}_{25}$. The operation is multiplication mod 25. Answer the following.:

   (a) How many elements does G contain? **Ans:** $\phi(25) = 25(4/5) = 20$.

   (b) List *all* the elements in the subgroup H generated by 11. These must all be explicit elements of $u(\mathbb{Z}_{25})$.

Ans: $11, 21, 6, 16, 1$

   (c) How many cosets of H are there in G? **Ans:** $20/5 = 4$.

9. The following is a *generating matrix* for a group code: $G = \left(\begin{array}{ccc|cccc} 1 & 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 & 1 & 0 \end{array}\right)$.

   (a) Use it to encode the following two messages: (101) and (011).

Ans: $(101)G = 101\,0101$, $(011)G = 011\,1001$

   (b) Write out the parity check matrix H associated with G.

Ans: $H = \left(\begin{array}{ccc|cccc} 1 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 1 \end{array}\right)$

   (c) Using H, correct (when necessary) the following two received words, assuming at most one bit is in error (or else explain why you think that cannot be done): (i) $r = (011\,0010)$, and (ii) $s = (011\,1000)$.

Ans: $Hr^{\text{tr}} = (1011)^{\text{tr}}$. Error in 1st position, $r \to (111\,0010)$
$Hs^{\text{tr}} = (0001)^{\text{tr}}$. Error in last position, $s \to (011\,1001)$

   (d) Write down the original message words for each of the received words in part (c).

Ans: For r: (111) and for s: (011)

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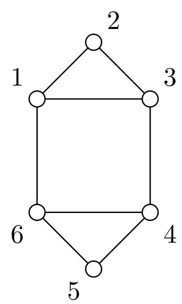
10. The questions below refer to the pictured geometric figure, consisting of a square with 2 identical, symmetrically placed isosceles triangles added. Please write all your answers in the large white space provided below.



(a) Write out the group of rigid motions of the figure, expressing each element as a permutation of the vertices using *disjoint cycle notation*.

(b) Find the number of distingushable colorings of the vertices if there are 5 colors to choose from. Do not simplify.

(c) Write out the cycle index polynomial for the group you found in part (a).

Ans: (a) Identity, rotation by 180°, and reflections in the horizontal and vertical axes of symmetry:

$$\{(1)(2)(3)(4)(5)(6), (14)(25)(36), (13)(2)(46)(5), (16)(25)(34)\}$$

(b) Average the number of colors to the power of the number of cycles:

$$\frac{1}{4}\left(5^6 + 5^3 + 5^3 + 5^4\right)$$

(c) Substitute x_1 for each 1-cycle and x_2 for each 2-cycle in each group element, then average those formulas:

$$\frac{1}{4}\left(x_1^6 + 2x_2^3 + x_1^2x_2^2\right)$$

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Math 3103 Combinatorics (Luecking) NAME:____________________
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Final Exam B(solutions) December 13, 2023

Note: If I ask for an answer "in elementary form", that means you must write it using only numbers and the operations of addition, subtraction, multiplication, division, powers and factorials. When I **do not** explicitly request this, you may also use $C(n,k)$, $\binom{n}{k}$, $P(n,k)$ and d_n (with explicit numbers, of course).

1. A contest has 6 prizes and 18 contestants. If all 6 prizes must be awarded to some contestant, how many possible ways could this be done under each of the following conditions. All these answers must be in elementary form.

   (a) The prizes are identical, and any contestant may receive any number of them.

Ans: Combination with repetition, 6 selections 18 people: $C(6+18-1,6) = \dfrac{23!}{6!\,(23-6)!}$

   (b) The prizes are all different, and any contestant may receive any number of them.

Ans: Select 6 times, with 18 possible choices each time: 18^6.

   (c) The prizes are all different, and no contestant may receive more than one.

Ans: Select 6 contestants and then endow each with a different prize: $P(18,6) = \dfrac{18!}{(18-6)!}$.

   (d) The prizes are identical, and no contestant may receive more than one.

Ans: Select a subset of size 6 from a set of size 18: $C(18,6) = \dfrac{18!}{6!(18-6)!}$.

2. Answer the following questions about the 9-letter string "NUTRITION", which has some repeated letters: 2 "N"s, 2 "T"s and 2 "I"s. The remaining letters occur only once each. All answers must be in elementary form.

   (a) How many different arrangements of this string are there? **Ans:** $\dfrac{9!}{2!\,2!\,2!}$

   (b) How many of the arrangements in part (a) contain **none** of the substrings "NN", "TT", "II"?

Ans: c_1 = 'contains "NN"'; c_2 = 'contains "TT"'; c_3 = 'contains "II"'.

$$S_1 = \binom{3}{1}N(c_1) = 3\left(\frac{8!}{2!\,2!}\right),\quad S_2 = \binom{3}{2}N(c_1c_2) = 3\left(\frac{7!}{2!}\right),\quad S_3 = N(c_1c_2c_3) = 6!, \text{ so}$$

$$N(\bar{c}_1\bar{c}_2\bar{c}_3) = N - S_1 + S_2 - S_3 = \frac{9}{2!\,2!\,2!} - 3\frac{8!}{2!\,2!} + 3\frac{7!}{2!} - 5!$$

   (c) How many arrangements of this string contain **at least one** of the three substrings "NN", "TT", "II"?

Ans: $L_1 = S_1 - S_2 + S_3 = 3\dfrac{8!}{2!\,2!} - 3\dfrac{7!}{2!} + 6!$. (Note: *Same* S_j as in part (b))

   (d) How many arrangements of this string contain **exactly one** of the three substrings "NN", "TT", "II"?

Ans: $E_1 = S_1 - 2S_2 + 3S_3 = 3\dfrac{8!}{2!\,2!} - 2\cdot 3\dfrac{7!}{2!} + 3\cdot 6!$. (Note: *Same* S_j as in part (b))

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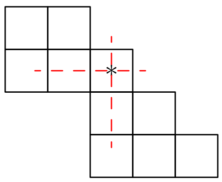
3. (a) Find the rook polynomial for the following chessboard C. Use the method that involves removing a square and the product formula. Write the answer as a sum of numbers times different powers of x, similar to the one in part (b).



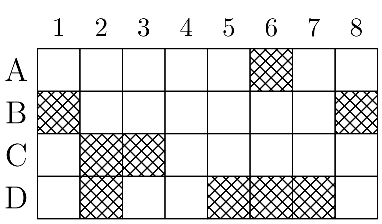
Ans: The first part of $r(C,x)$ below comes from removing the starred square, the second part from removing all the squares in the same row and column as the starred one.

$$r(C,x) = (1+4x+2x^2)(1+5x+4x^2) + x(1+2x)(1+3x+x^2)$$
$$= 1+10x+31x^2+33x^3+10x^4 \,.$$

(b) This table represents possible seating of 4 people among 8 seats. A shaded square represents a forbidden seat assignment. The rook polynomial of the shaded squares is

$$1+9x+26x^2+29x^3+10x^4$$

Using this information, how many allowable ways are there to seat all five people? This answer must be in elementary form.



Ans: $P(8,4) - 9P(7,3) + 26P(6,2) - 29P(5,1) + 10P(4,0)$

$$= \frac{8!}{4!} - 9\left(\frac{7!}{4!}\right) + 26\left(\frac{6!}{4!}\right) - 29\left(\frac{5!}{4!}\right) + 10\left(\frac{4!}{4!}\right)$$

Don't write in this box

4. Consider the following equation. The variables w_1, w_2 and w_3 are integers that must satisfy the given conditions.

$$w_1 + w_2 + w_3 = n$$

$$10 \le w_1 \longrightarrow x^{10} + x^9 + \cdots = \frac{x^{10}}{1-x}$$

$$0 \le w_2 \le 19 \longrightarrow 1 + x + x^2 + \cdots + x^{19} = \frac{1-x^{20}}{1-x}$$

$$10 \le w_3 \le 29 \longrightarrow x^{10} + x^2 + \cdots + x^{29} = \frac{x^{10}-x^{30}}{1-x}$$

(a) Write out the generating function for the number of solutions of this equation. Write your answer as a quotient in which the denominator is a power of $(1-x)$ and the numerator is a polynomial written out as a sum of powers of x.

Ans: Take the product of the above fractions: $F(x) = \dfrac{x^{20} - 2x^{40} + x^{60}}{(1-x)^3}$.

(b) Use the result in part (a) to find the number of solutions when $n = 100$.

Ans: $F(x) = \left(x^{20} - 2x^{40} + x^{60}\right) \displaystyle\sum_{n=0}^{\infty} \binom{n+3-1}{n} x^n$, and the terms that produce x^{100} are:

$$x^{20}\binom{82}{80}x^{80} - 2x^{40}\binom{62}{60}x^{60} + x^{60}\binom{42}{40}x^{40} \text{ leading to } \binom{82}{80} - 2\binom{62}{60} + \binom{42}{40}$$

5. For the following second-order recurrence relations, find (i) the characteristic equation and its roots, (ii) the general solution, and (iii) the solution that satisfies the initial conditions.

(a) $a_n - 5a_{n-1} + 4a_{n-2} = 0\,,\ \ n \ge 2\,,$
$a_0 = 0$ and $a_1 = 6,$.

Ans: (i) $r^2 - 5r + 4 = 0$, roots 1 and 4. (ii) General solution: $C_1 + C_2 4^n$. (iii) The ICs, $C_1 + C_2 = 0$ and $C_1 + 4C_2 = 6$, give $C_1 = -2$ and $C_2 = 2$. Thus: $a_n = -2 + 2 \cdot 4^n$

(b) $a_n - 6a_{n-1} + 13a_{n-2} = 0\,,\ \ n \ge 2\,,$
$a_0 = 0$ and $a_1 = 5,$.

Ans: (i) $r^2 - 6r + 13 = 0$, roots $3 \pm 2i$. (ii) General solution: $C_1(3+2i)^n + C_2(3-2i)^n$. (iii) The ICs, $C_1 + C_2 = 0$ and $3(C_1 + C_2) + 2i(C_1 - C_2) = 5$, give $C_1 = \dfrac{5}{4i}$ and $C_2 = -\dfrac{5}{4i}$. Thus: $a_n = \dfrac{5}{4i}(3+2i)^n - \dfrac{5}{4i}(3-2i)^n$

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6. For the following nonhomogeneous recurrence relation with initial conditions, find (i) the homogeneous solution $a_n^{(h)}$, (ii) a particular solution $a_n^{(p)}$, and (iii) the solution that satisfies the initial conditions.

$a_n - 4a_{n-1} + 4a_{n-2} = (2/9)3^n\,,\quad n \geq 2\,,$

$a_0 = 0 \quad \text{and} \quad a_1 = 0\,.$

Ans: (i) The homogeneous solution is $a_n^{(h)} = C_1 2^n + C_2 n 2^n$.

(ii) A particular solution exists in the form $a_n^{(p)} = A3^n$ for some constant A. The recurrence relation gives $A - (4/3)A + (4/9)A = 2/9$, so $A = 2$ and $a_n^{(p)} = (2)3^n$.

(iii) The general solution is therefore $C_1 2^n + C_2 n 2^n + (2)3^n$. The ICs, $C_1 + 2 = 0$ and $2C_1 + 2C_2 + 6 = 0$, give $C_1 = -2$ and $C_2 = -1$. Thus:

$$a_n = -(2)2^n - n2^n + (2)3^n$$

7. The factorization of 2125 into primes is $2125 = 5^3(17)$ Do the following for the ring $\mathbb{Z}_{2125}$. Your final answers in parts (a) and (b) must **contain no fractions**.

(a) Find the number of units.

Ans: $\phi(2125) = 5^3(17)\left(\frac{4}{5}\right)\left(\frac{16}{17}\right) = 5^2(4)(16) = 1600$.

(b) Find the number of proper zero divisors.

Ans: $2125 - 1600 - 1 = 524$.

(c) Using the Euclidean algorithm, find $(101)^{-1}$ in this ring. Your answer must be an explicit element of $\mathbb{Z}_{2125}$.

Ans: The Euclidean algorithm gives:

$$\begin{cases} 2125 = 21(101) + 4 \\ \ \ 101 = \quad 25(4) + 1 \end{cases} \quad \text{or} \quad \begin{cases} n = 21k \ + r_1 \\ k = 25r_1 + r_2 \end{cases}$$

where $n = 2125$, $k = 101$, $r_1 = 4$, and $r_2 = 1$. Putting $r_1 = n - 21k$ (from the top equation) into the bottom one gives
$k = 25n - 525k + r_2$ and solving for r_2: $r_2 = 526k - 25n$. If we put the values back in, this says $1 = 526 \cdot 101$ in the ring $\mathbb{Z}_{2125}$. Thus $(101)^{-1} = 526$.

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8. Let $G = u(\mathbb{Z}_{25})$, the group of units of the ring $\mathbb{Z}_{25}$. The operation is multiplication mod 25. Answer the following.:

   (a) How many elements does G contain? **Ans:** $\phi(25) = 25(4/5) = 20$.

   (b) List *all* the elements in the subgroup H generated by 7. These must all be explicit elements of $u(\mathbb{Z}_{25})$.

Ans: $7, 24, 18, 1$

   (c) How many cosets of H are there in G? **Ans:** $20/4 = 5$.

9. The following is a *generating matrix* for a group code: $G = \left(\begin{array}{ccc|cccc} 1 & 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 & 1 \end{array}\right)$.

   (a) Use it to encode the following two messages: (101) and (011).

Ans: $(101)G = 101\,1100$, $(011)G = 011\,1001$

   (b) Write out the parity check matrix H associated with G.

Ans: $H = \left(\begin{array}{ccc|cccc} 1 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 1 \end{array}\right)$

   (c) Using H, correct (when necessary) the following two received words, assuming at most one bit is in error (or else explain why you think that cannot be done): (i) $r = (011\,0010)$, and (ii) $s = (011\,1000)$.

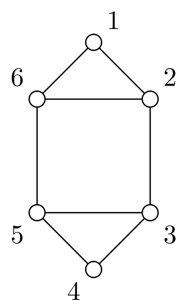
Ans: $Hr^{\text{tr}} = (1011)^{\text{tr}}$. Error in 1st position, $r \to (111\,0010)$
$Hs^{\text{tr}} = (0001)^{\text{tr}}$. Error in last position, $s \to (011\,1001)$

   (d) Write down the original message words for each of the received words in part (c).

Ans: For r: (111) and for s: (011)

Don't write in this box

10. The questions below refer to the pictured geometric figure, consisting of a square with 2 identical, symmetrically placed isosceles triangles added. Please write all your answers in the large white space provided below.



(a) Write out the group of rigid motions of the figure, expressing each element as a permutation of the vertices using *disjoint cycle notation.*

(b) Find the number of distingushable colorings of the vertices if there are 6 colors to choose from. Do not simplify.

(c) Write out the cycle index polynomial for the group you found in part (a).

Ans: (a) Identity, rotation by 180°, and reflections in the horizontal and vertical axes of symmetry:

$$\{(1)(2)(3)(4)(5)(6), (14)(25)(36), (1)(26)(35)(4), (14)(23)(56)\}$$

(b) Average the number of colors to the power of the number of cycles:

$$\frac{1}{4}\left(6^6 + 6^3 + 6^3 + 6^4\right)$$

(c) Substitute x_1 for each 1-cycle and x_2 for each 2-cycle in each group element, then average those formulas:

$$\frac{1}{4}\left(x_1^6 + 2x_2^3 + x_1^2x_2^2\right)$$

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Math 3103 Combinatorics (Luecking) NAME:______________________
(Please print clearly)

Final Exam C(solutions) December 13, 2023

Note: If I ask for an answer "in elementary form", that means you must write it using only numbers and the operations of addition, subtraction, multiplication, division, powers and factorials. When I **do not** explicitly request this, you may also use $C(n,k)$, $\binom{n}{k}$, $P(n,k)$ and d_n (with explicit numbers, of course).

1. A contest has 7 prizes and 19 contestants. If all 7 prizes must be awarded to some contestant, how many possible ways could this be done under each of the following conditions. All these answers must be in elementary form.

   (a) The prizes are identical, and any contestant may receive any number of them.

   Ans: Combination with repetition, 7 selections 19 people: $C(7+19-1,7) = \dfrac{25!}{7!\,(25-7)!}$

   (b) The prizes are all different, and any contestant may receive any number of them.

   Ans: Select 7 times, with 19 possible choices each time: 19^7.

   (c) The prizes are all different, and no contestant may receive more than one.

   Ans: Select 7 contestants and then endow each with a different prize: $P(19,7) = \dfrac{19!}{(19-7)!}$.

   (d) The prizes are identical, and no contestant may receive more than one.

   Ans: Select a subset of size 7 from a set of size 19: $C(19,7) = \dfrac{19!}{7!(19-7)!}$.

2. Answer the following questions about the 8-letter string "PORPHYRY", which has some repeated letters: 2 "P"s, 2 "R"s and 2 "Y"s. The remaining letters occur only once each. All answers must be in elementary form.

   (a) How many different arrangements of this string are there? **Ans:** $\dfrac{8!}{2!\,2!\,2!}$

   (b) How many of the arrangements in part (a) contain **none** of the substrings "PP", "RR", "YY"?

   Ans: $c_1 =$ 'contains "PP"'; $c_2 =$ 'contains "RR"'; $c_3 =$ 'contains "YY"'.

   $S_1 = \binom{3}{1} N(c_1) = 3\left(\dfrac{7!}{2!\,2!}\right)$, $S_2 = \binom{3}{2} N(c_1 c_2) = 3\left(\dfrac{6!}{2!}\right)$, $S_3 = N(c_1 c_2 c_3) = 5!$, so

   $$N(\bar{c}_1 \bar{c}_2 \bar{c}_3) = N - S_1 + S_2 - S_3 = \frac{8}{2!\,2!\,2!} - 3\frac{7!}{2!\,2!} + 3\frac{6!}{2!} - 5!$$

   (c) How many arrangements of this string contain **at least one** of the three substrings "PP", "RR", "YY"?

   Ans: $L_1 = S_1 - S_2 + S_3 = 3\dfrac{7!}{2!\,2!} - 3\dfrac{6!}{2!} + 5!$. (Note: *Same* S_j as in part (b))

   (d) How many arrangements of this string contain **exactly one** of the three substrings "PP", "RR", "YY"?

   Ans: $E_1 = S_1 - 2S_2 + 3S_3 = 3\dfrac{7!}{2!\,2!} - 2\cdot 3\dfrac{6!}{2!} + 3\cdot 5!$. (Note: *Same* S_j as in part (b))

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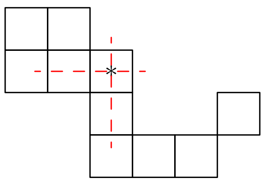
3. (a) Find the rook polynomial for the following chessboard C. Use the method that involves removing a square and the product formula. Write the answer as a sum of numbers times different powers of x, similar to the one in part (b).



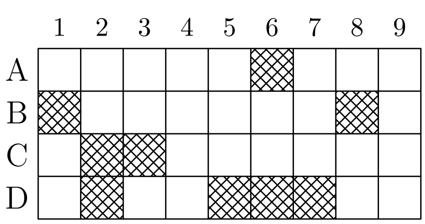
Ans: The first part of $r(C,x)$ below comes from removing the starred square, the second part from removing all the squares in the same row and column as the starred one.

$$r(C,x) = (1+4x+2x^2)(1+5x+5x^2) + x(1+2x)(1+3x+2x^2)$$
$$= 1+10x+32x^2+38x^3+14x^4\,.$$

(b) This table represents possible seating of 4 people among 9 seats. A shaded square represents a forbidden seat assignment. The rook polynomial of the shaded squares is

$$1+9x+26x^2+29x^3+10x^4$$

Using this information, how many allowable ways are there to seat all five people? This answer must be in elementary form.



Ans: $P(9,4) - 9P(8,3) + 26P(7,2) - 29P(6,1) + 10P(5,0)$

$$= \frac{9!}{5!} - 9\left(\frac{8!}{5!}\right) + 26\left(\frac{7!}{5!}\right) - 29\left(\frac{6!}{5!}\right) + 10\left(\frac{5!}{5!}\right)$$

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4. Consider the following equation. The variables w_1, w_2 and w_3 are integers that must satisfy the given conditions.

$$w_1 + w_2 + w_3 = n$$

$$8 \le w_1 \longrightarrow x^8 + x^9 + \cdots = \frac{x^8}{1-x}$$

$$0 \le w_2 \le 19 \longrightarrow 1 + x + x^2 + \cdots + x^{19} = \frac{1-x^{20}}{1-x}$$

$$2 \le w_3 \le 21 \longrightarrow x^2 + x^2 + \cdots + x^{21} = \frac{x^2 - x^{22}}{1-x}$$

(a) Write out the generating function for the number of solutions of this equation. Write your answer as a quotient in which the denominator is a power of $(1-x)$ and the numerator is a polynomial written out as a sum of powers of x.

Ans: Take the product of the above fractions: $F(x) = \dfrac{x^{10} - 2x^{30} + x^{50}}{(1-x)^3}$.

(b) Use the result in part (a) to find the number of solutions when $n = 100$.

Ans: $F(x) = \left(x^{10} - 2x^{30} + x^{50}\right) \displaystyle\sum_{n=0}^{\infty} \binom{n+3-1}{n} x^n$, and the terms that produce x^{100} are:

$$x^{10}\binom{92}{90}x^{90} - 2x^{30}\binom{72}{70}x^{70} + x^{50}\binom{52}{50}x^{50} \text{ leading to } \binom{92}{90} - 2\binom{72}{70} + \binom{52}{50}$$

5. For the following second-order recurrence relations, find (i) the characteristic equation and its roots, (ii) the general solution, and (iii) the solution that satisfies the initial conditions.

(a) $a_n - 6a_{n-1} + 5a_{n-2} = 0\,,\ \ n \ge 2\,,$
$a_0 = 0$ and $a_1 = 8,.$

Ans: (i) $r^2 - 6r + 5 = 0$, roots 1 and 5. (ii) General solution: $C_1 + C_2 5^n$. (iii) The ICs, $C_1 + C_2 = 0$ and $C_1 + 5C_2 = 8$, give $C_1 = -2$ and $C_2 = 2$. Thus: $a_n = -2 + 2 \cdot 5^n$

(b) $a_n - 4a_{n-1} + 13a_{n-2} = 0\,,\ \ n \ge 2\,,$
$a_0 = 0$ and $a_1 = 7,.$

Ans: (i) $r^2 - 4r + 13 = 0$, roots $2 \pm 3i$. (ii) General solution: $C_1(2+3i)^n + C_2(2-3i)^n$. (iii) The ICs, $C_1 + C_2 = 0$ and $2(C_1 + C_2) + 3i(C_1 - C_2) = 7$, give $C_1 = \dfrac{7}{6i}$ and $C_2 = -\dfrac{7}{6i}$. Thus: $a_n = \dfrac{7}{6i}(2+3i)^n - \dfrac{7}{6i}(2-3i)^n$

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6. For the following nonhomogeneous recurrence relation with initial conditions, find (i) the homogeneous solution $a_n^{(h)}$, (ii) a particular solution $a_n^{(p)}$, and (iii) the solution that satisfies the initial conditions.

$$a_n - 4a_{n-1} + 4a_{n-2} = (4/9)3^n\,,\quad n \geq 2\,,$$
$$a_0 = 0 \quad \text{and} \quad a_1 = 0\,.$$

Ans: (i) The homogeneous solution is $a_n^{(h)} = C_1 2^n + C_2 n 2^n$.

(ii) A particular solution exists in the form $a_n^{(p)} = A3^n$ for some constant A. The recurrence relation gives $A - (4/3)A + (4/9)A = 4/9$, so $A = 4$ and $a_n^{(p)} = (4)3^n$.

(iii) The general solution is therefore $C_1 2^n + C_2 n 2^n + (4)3^n$. The ICs, $C_1 + 4 = 0$ and $2C_1 + 2C_2 + 12 = 0$, give $C_1 = -4$ and $C_2 = -2$. Thus:

$$a_n = -(4)2^n - (2)n2^n + (4)3^n$$

7. The factorization of 2133 into primes is $2133 = 3^3(79)$ Do the following for the ring $\mathbb{Z}_{2133}$. Your final answers in parts (a) and (b) must **contain no fractions**.

(a) Find the number of units.

Ans: $\phi(2133) = 3^3(79)\left(\frac{2}{3}\right)\left(\frac{78}{79}\right) = 3^2(2)(78) = 1404$.

(b) Find the number of proper zero divisors.

Ans: $2133 - 1404 - 1 = 728$.

(c) Using the Euclidean algorithm, find $(100)^{-1}$ in this ring. Your answer must be an explicit element of $\mathbb{Z}_{2133}$.

Ans: The Euclidean algorithm gives:

$$\begin{cases} 2133 = 21(100) + 33 \\ 100 = 3(33) + 1 \end{cases} \quad \text{or} \quad \begin{cases} n = 21k + r_1 \\ k = 3r_1 + r_2 \end{cases}$$

where $n = 2133$, $k = 100$, $r_1 = 33$, and $r_2 = 1$. Putting $r_1 = n - 21k$ (from the top equation) into the bottom one gives

$k = 3n - 63k + r_2$ and solving for r_2: $r_2 = 64k - 3n$. If we put the values back in, this says $1 = 64 \cdot 100$ in the ring $\mathbb{Z}_{2133}$. Thus $(100)^{-1} = 64$.

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8. Let $G = u(\mathbb{Z}_{25})$, the group of units of the ring $\mathbb{Z}_{25}$. The operation is multiplication mod 25. Answer the following.:

(a) How many elements does G contain? **Ans:** $\phi(25) = 25(4/5) = 20$.

(b) List *all* the elements in the subgroup H generated by 11. These must all be explicit elements of $u(\mathbb{Z}_{25})$.

Ans: $11, 21, 6, 16, 1$

(c) How many cosets of H are there in G? **Ans:** $20/5 = 4$.

9. The following is a *generating matrix* for a group code: $G = \left(\begin{array}{ccc|cccc} 1 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 & 1 & 0 \end{array}\right)$.

(a) Use it to encode the following two messages: (101) and (011).

Ans: $(101)G = 101\,1001$, $(011)G = 011\,0101$

(b) Write out the parity check matrix H associated with G.

Ans: $H = \left(\begin{array}{ccc|cccc} 0 & 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 1 \end{array}\right)$

(c) Using H, correct (when necessary) the following two received words, assuming at most one bit is in error (or else explain why you think that cannot be done): (i) $r = (011\,0010)$, and (ii) $s = (011\,0001)$.

Ans: $Hr^{\text{tr}} = (0111)^{\text{tr}}$. Error in 1st position, $r \to (111\,0010)$

$Hs^{\text{tr}} = (0100)^{\text{tr}}$. Error in fifth position, $s \to (011\,0101)$

(d) Write down the original message words for each of the received words in part (c).

Ans: For r: (111) and for s: (011)

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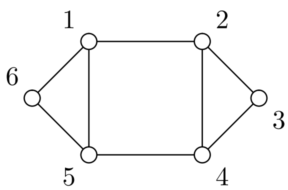
10. The questions below refer to the pictured geometric figure, consisting of a square with 2 identical, symmetrically placed isosceles triangles added. Please write all your answers in the large white space provided below.



(a) Write out the group of rigid motions of the figure, expressing each element as a permutation of the vertices using *disjoint cycle notation*.

(b) Find the number of distingushable colorings of the vertices if there are 7 colors to choose from. Do not simplify.

(c) Write out the cycle index polynomial for the group you found in part (a).

Ans: (a) Identity, rotation by 180°, and reflections in the horizontal and vertical axes of symmetry:

$$\{(1)(2)(3)(4)(5)(6), (14)(25)(36), (12)(36)(45), (15)(24)(3)(6)\}$$

(b) Average the number of colors to the power of the number of cycles:

$$\frac{1}{4}\left(7^6 + 7^3 + 7^3 + 7^4\right)$$

(c) Substitute x_1 for each 1-cycle and x_2 for each 2-cycle in each group element, then average those formulas:

$$\frac{1}{4}\left(x_1^6 + 2x_2^3 + x_1^2x_2^2\right)$$

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