Math 3103 Combinatorics (Luecking) NAME:____________________________
(Please print clearly)

Third Exam A December 1, 2023

1. Let $G = \mathcal{S}_5$, the group of all permutations of the set $\{1, 2, 3, 4, 5\}$. The operation is composition of permutations. Answer the following:

   (a) How many elements does G contain? **Ans:** $5! = 120$.

   (b) $\alpha = (12)(345)$ is an element of G written in disjoint cycle notation. List *all* the elements in the subgroup H generated by α. Express all the elements in disjoint cycle notation.

   Ans: $(12)(345), (1)(2)(354), (12)(3)(4)(5), (1)(2)(345), (12)(354), (1)(2)(3)(4)(5)$

   (c) How many cosets of H are there in G? **Ans:** $5!/6 = 20$.

2. Let $G = u(\mathbb{Z}_{50})$, the group of units of the ring $\mathbb{Z}_{50}$ with the operation of multiplication mod 50. Answer the following:

   (a) How many elements does G contain? **Ans:** $\phi(50) = 50(1 - 1/2)(1 - 1/5) = 20$.

   (b) List all the elements in the cyclic subgroup H generated by the element 11.

   Ans: $11, 21, 31, 41, 1$

   (c) How many cosets of H are there in G? **Ans:** $20/5 = 4$.

Don't write in this box

3. For each of the following *group codes*, (i) determine the minimum distance between code words; (ii) determine the maximum number of errors that can be reliably *detected*, and (iii) determine the maximum number of errors that can be reliably *corrected*.

(a) $C_1 = \{(000\,000\,00),\ (100\,110\,10),\ (010\,101\,01),\ (001\,011\,10),$
$(110\,011\,11),\ (011\,110\,11),\ (101\,101\,00),\ (111\,000\,01)\}$ in $\mathbb{Z}_2^8$.

Ans: (i) 4, (ii) 3, (iii) 1.

(b) $C_2 = \{(000\,000\,0000),\ (100\,111\,1000),\ (010\,001\,1110),\ (001\,010\,1011),$
$(110\,110\,0110),\ (011\,011\,0101),\ (101\,101\,0011),\ (111\,100\,1101)\}$ in $\mathbb{Z}_2^{10}$.

Ans: (i) 5, (ii) 4, (iii) 2.

4. The following is a *generating matrix* for a group code: $G = \left(\begin{array}{cccc|ccc} 1 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 0 \end{array}\right)$.

(a) Use it to encode the following two messages: (1011) and (0110).

Ans: $(1011)G = 1011\,010$, $(0110)G = 0110\,010$

(b) Write out the parity check matrix H associated with G.

Ans: $H = \left(\begin{array}{cccc|ccc} 0 & 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 & 1 \end{array}\right)$

(c) Using H, correct the following two received words, assuming at most one bit is in error (or else explain why you think that cannot be done):
(i) $r = (1100\,001)$, and (ii) $s = (1110\,010)$.

Ans: $Hr^{\text{tr}} = (111)^{\text{tr}}$, change bit 3 to get $(1110\,001)$
$Hs^{\text{tr}} = (011)^{\text{tr}}$, change bit 1 to get $(0110\,010)$

(d) Write down the original message words for each of the received words in part (c).

Ans: For r: (1110) and for s: (0110)

Don't write in this box

5. For each of the two geometric figures below, write out the group of all rigid motions of the figure. Write the elements as permutations of the vertex labels, using disjoint cycle notation.

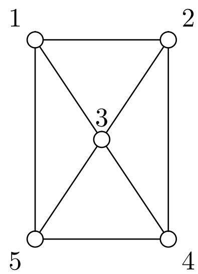
(a) 

Figure A:
Rectangle with diagonals and center vertex added

Ans: Identity, 180° rotation, left-right and top-bottom reflection
$\{(1)(2)(3)(4)(5), (14)(25)(3), (12)(45)(3), (15)(24)(3)\}$

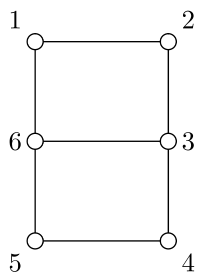
(b) 

Figure B:
Rectangle with midpoints of two sides added, plus a line connecting them

Ans: Identity, 180° rotation, left-right and top-bottom reflection
$\{(1)(2)(3)(4)(5)(6), (14)(25)(36), (12)(36)(45), (15)(24)(3)(6)\}$

6. For the two figures in problem 5 do the following. Do not simplify the resulting expressions.

(a) Figure A: determine the number of distinguishable colorings of the vertices when there are 4 colors to choose from.

Ans: $\frac{1}{4}(4^5 + 4^3 + 4^3 + 4^3)$

(b) Figure B: determine the number of distinguishable colorings of the vertices when there are 6 colors to choose from.

Ans: $\frac{1}{4}(6^6 + 6^3 + 6^3 + 6^4)$

(c) Write down the cycle index polynomial for the group you found for Figure A.

Ans: $\frac{1}{4}\left(x_1^5 + 3x_1x_2^2\right)$

Don't write in this box

Math 3103 Combinatorics (Luecking) NAME:__________________________
(Please print clearly)

Third Exam B December 1, 2023

1. Let $G = \mathcal{S}_5$, the group of all permutations of the set $\{1, 2, 3, 4, 5\}$. The operation is composition of permutations. Answer the following:

   (a) How many elements does G contain? **Ans:** $5! = 120$.

   (b) $\alpha = (123)(45)$ is an element of G written in disjoint cycle notation. List *all* the elements in the subgroup H generated by α. Express all the elements in disjoint cycle notation.

Ans: $(123)(45), (132)(4)(5), (1)(2)(3)(45), (123)(4)(5), (132)(45), (1)(2)(3)(4)(5)$

   (c) How many cosets of H are there in G? **Ans:** $5!/6 = 20$.

2. Let $G = u(\mathbb{Z}_{50})$, the group of units of the ring $\mathbb{Z}_{50}$ with the operation of multiplication mod 50. Answer the following:

   (a) How many elements does G contain? **Ans:** $\phi(50) = 50(1 - 1/2)(1 - 1/5) = 20$.

   (b) List all the elements in the cyclic subgroup H generated by the element 7.

Ans: $7, 49, 43, 1$

   (c) How many cosets of H are there in G? **Ans:** $20/4 = 5$.

Don't write in this box

3. For each of the following *group codes*, (i) determine the minimum distance between code words; (ii) determine the maximum number of errors that can be reliably *detected*, and (iii) determine the maximum number of errors that can be reliably *corrected*.

   (a) $C_1 = \{(000\,000\,0000),\ (100\,111\,1000),\ (010\,001\,1110),\ (001\,010\,1011),$
   $(110\,110\,0110),\ (011\,011\,0101),\ (101\,101\,0011),\ (111\,100\,1101)\}$ in $\mathbb{Z}_2^{10}$.

Ans: (i) 5, (ii) 4, (iii) 2.

   (b) $C_2 = \{(000\,000\,00),\ (100\,110\,10),\ (010\,101\,01),\ (001\,011\,10),$
   $(110\,011\,11),\ (011\,110\,11),\ (101\,101\,00),\ (111\,000\,01)\}$ in $\mathbb{Z}_2^{8}$.

Ans: (i) 4, (ii) 3, (iii) 1.

4. The following is a *generating matrix* for a group code: $G = \left(\begin{array}{cccc|ccc} 1 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 0 \end{array}\right)$.

   (a) Use it to encode the following two messages: (1011) and (0110).

Ans: $(1011)G = 1011\,100$, $(0110)G = 0110\,110$

   (b) Write out the parity check matrix H associated with G.

Ans: $H = \left(\begin{array}{cccc|ccc} 1 & 0 & 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 & 1 \end{array}\right)$

   (c) Using H, correct the following two received words, assuming at most one bit is in error (or else explain why you think that cannot be done):
   (i) $r = (1100\,001)$, and (ii) $s = (1110\,010)$.

Ans: $Hr^{\text{tr}} = (101)^{\text{tr}}$, change bit 3 to get $(1110\,001)$
$Hs^{\text{tr}} = (011)^{\text{tr}}$, change bit 2 to get $(1010\,010)$

   (d) Write down the original message words for each of the received words in part (c).

Ans: For r: (1110) and for s: (1010)

Don't write in this box

5. For each of the two geometric figures below, write out the group of all rigid motions of the figure. Write the elements as permutations of the vertex labels, using disjoint cycle notation.

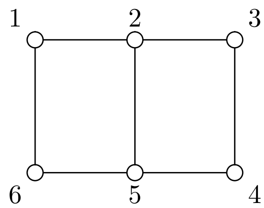
(a) 

Figure A:
Rectangle with midpoints of two sides added, plus a line connecting them

Ans: Identity, 180° rotation, left-right and top-bottom reflection

$\{(1)(2)(3)(4)(5)(6), (14)(25)(36), (13)(2)(46)(5), (16)(25)(34)\}$

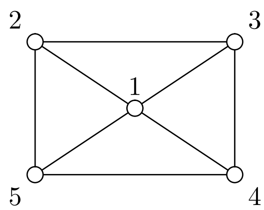
(b) 

Figure B:
Rectangle with diagonals and center vertex added

Ans: Identity, 180° rotation, left-right and top-bottom reflection

$\{(1)(2)(3)(4)(5), (1)(24)(35), (1)(23)(45), (1)(25)(34)\}$

6. For the two figures in problem 5 do the following. Do not simplify the resulting expressions.

(a) Figure A: determine the number of distinguishable colorings of the vertices when there are 4 colors to choose from.

Ans: $\frac{1}{4}(4^6 + 4^3 + 4^4 + 4^3)$

(b) Figure B: determine the number of distinguishable colorings of the vertices when there are 6 colors to choose from.

Ans: $\frac{1}{4}(6^5 + 6^3 + 6^3 + 6^3)$

(c) Write down the cycle index polynomial for the group you found for Figure A.

Ans: $\frac{1}{4}\left(x_1^6 + 2x_2^3 + x_1^2 x_2^2\right)$

Don't write in this box

Math 3103 Combinatorics (Luecking) NAME:____________________
(Please print clearly)

Third Exam C December 1, 2023

1. Let $G = \mathcal{S}_5$, the group of all permutations of the set $\{1, 2, 3, 4, 5\}$. The operation is composition of permutations. Answer the following:

   (a) How many elements does G contain? **Ans:** $5! = 120$.

   (b) $\alpha = (12)(345)$ is an element of G written in disjoint cycle notation. List *all* the elements in the subgroup H generated by α. Express all the elements in disjoint cycle notation.

Ans: $(12)(345), (1)(2)(354), (12)(3)(4)(5), (1)(2)(345), (12)(354), (1)(2)(3)(4)(5)$

   (c) How many cosets of H are there in G? **Ans:** $5!/6 = 20$.

2. Let $G = u(\mathbb{Z}_{50})$, the group of units of the ring $\mathbb{Z}_{50}$ with the operation of multiplication mod 50. Answer the following:

   (a) How many elements does G contain? **Ans:** $\phi(50) = 50(1 - 1/2)(1 - 1/5) = 20$.

   (b) List all the elements in the cyclic subgroup H generated by the element 7.

Ans: $7, 49, 43, 1$

   (c) How many cosets of H are there in G? **Ans:** $20/4 = 5$.

Don't write in this box

3. For each of the following *group codes*, (i) determine the minimum distance between code words; (ii) determine the maximum number of errors that can be reliably *detected*, and (iii) determine the maximum number of errors that can be reliably *corrected*.

(a) $C_1 = \{(000\,000\,00),\ (100\,110\,10),\ (010\,101\,01),\ (001\,011\,10),$
$(110\,011\,11),\ (011\,110\,11),\ (101\,101\,00),\ (111\,000\,01)\}$ in $\mathbb{Z}_2^8$.

Ans: (i) 4, (ii) 3, (iii) 1.

(b) $C_2 = \{(000\,000\,0000),\ (100\,111\,1000),\ (010\,001\,1110),\ (001\,010\,1011),$
$(110\,110\,0110),\ (011\,011\,0101),\ (101\,101\,0011),\ (111\,100\,1101)\}$ in $\mathbb{Z}_2^{10}$.

Ans: (i) 5, (ii) 4, (iii) 2.

4. The following is a *generating matrix* for a group code: $G = \left(\begin{array}{cccc|ccc} 1 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 0 \end{array}\right)$.

(a) Use it to encode the following two messages: (1011) and (0110).

Ans: $(1011)G = 1011\,100$, $(0110)G = 0110\,110$

(b) Write out the parity check matrix H associated with G.

Ans: $H = \left(\begin{array}{cccc|ccc} 1 & 0 & 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 & 1 \end{array}\right)$

(c) Using H, correct the following two received words, assuming at most one bit is in error (or else explain why you think that cannot be done):
(i) $r = (1100\,001)$, and (ii) $s = (1110\,010)$.

Ans: $Hr^{\text{tr}} = (101)^{\text{tr}}$, change bit 3 to get $(1110\,001)$
$Hs^{\text{tr}} = (011)^{\text{tr}}$, change bit 2 to get $(1010\,010)$

(d) Write down the original message words for each of the received words in part (c).

Ans: For r: (1110) and for s: (1010)

Don't write in this box

5. For each of the two geometric figures below, write out the group of all rigid motions of the figure. Write the elements as permutations of the vertex labels, using disjoint cycle notation.

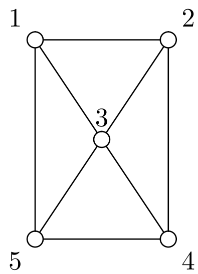
(a) 

Figure A:
Rectangle with diagonals and center vertex added

Ans: Identity, 180° rotation, left-right and top-bottom reflection

$\{(1)(2)(3)(4)(5), (14)(25)(3), (12)(45)(3), (15)(24)(3)\}$

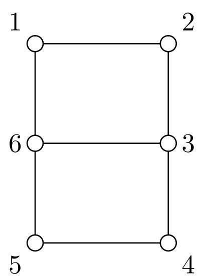
(b) 

Figure B:
Rectangle with midpoints of two sides added, plus a line connecting them

Ans: Identity, 180° rotation, left-right and top-bottom reflection

$\{(1)(2)(3)(4)(5)(6), (14)(25)(36), (12)(36)(45), (15)(24)(3)(6)\}$

6. For the two figures in problem 5 do the following. Do not simplify the resulting expressions.

   (a) Figure A: determine the number of distinguishable colorings of the vertices when there are 6 colors to choose from.

Ans: $\frac{1}{4}(6^5 + 6^3 + 6^3 + 6^3)$

   (b) Figure B: determine the number of distinguishable colorings of the vertices when there are 4 colors to choose from.

Ans: $\frac{1}{4}(4^6 + 4^3 + 4^3 + 4^4)$

   (c) Write down the cycle index polynomial for the group you found for Figure A.

Ans: $\frac{1}{4}\left(x_1^5 + 3x_1x_2^2\right)$

Don't write in this box