Math 3103 Combinatorics (Luecking) NAME:______________________________
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Second Exam A (solutions) October 27, 2023

1. For each of the following first-order recurrence relations, find the solution that satisfies the given initial condition.

   (a) $a_n = 3a_{n-1}\,,\;\; n \geq 1\,,$
   $a_0 = 10\,.$

Ans: Geometric progression: $a_n = (10)3^n$

   (b) $a_n = a_{n-1} + 4\,,\;\; n \geq 1\,,$
   $a_0 = 7\,.$

Ans: Arithmetic progression: $a_n = 7 + 4n$

   (c) $a_n = (n^2 + 4)a_{n-1}\,,\;\; n \geq 1\,,$
   $a_0 = 5\,.$

Ans: Successive multiplications: $a_n = 5(1^2 + 4)(2^2 + 4) \cdots (n^2 + 4)$.

   (d) $a_n = a_{n-1} + n^3 4^n\,,\;\; n \geq 1\,,$
   $a_0 = 2\,.$

Ans: Successive additions: $a_n = 2 + (1^3)4^1 + (2^3)4^2 + \cdots + n^3 4^n$

2. For the following second-order recurrence relation, find (i) the characteristic equation and its roots, (ii) the general solution, and (iii) the solution that satisfies the initial conditions.

$$a_n - 6a_{n-1} + 9a_{n-2} = 0\,,\;\; n \geq 2\,,$$
$$a_0 = 1 \;\text{ and }\; a_1 = 4,.$$

Ans: (i) $r^2 - 6r + 9 = 0$ has roots 3 and 3 (repeated root).

(ii) General solution: $C_1 3^n + C_2 n 3^n$.

(iii) The ICs, $C_1 = 1$ and $3C_1 + 3C_2 = 4$, give $C_1 = 1$ and $C_2 = 1/3$. Thus:

$$a_n = 3^n + \frac{1}{3} n 3^n$$

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3. For the following second-order recurrence relation, find (i) the characteristic equation and its roots, (ii) the general solution, and (iii) the solution that satisfies the initial conditions.

$$a_n - 4a_{n-1} + 5a_{n-2} = 0\,,\quad n \geq 2\,,$$
$$a_0 = 0 \quad\text{and}\quad a_1 = 4.$$

Ans: (i) $r^2 - 4r + 5 = 0$ has roots $r = 2 \pm i$.

(ii) General solution: $C_1(2+i)^n + C_2(2-i)^n$.

(iii) The ICs, $C_1 + C_2 = 0$ and $C_1(2+i) + C_2(2-i) = 4$, give $C_1 = -2i$ and $C_2 = 2i$.

Thus: $a_n = -2i(2+i)^n + 2i(2-i)^n$.

Or use the form $a_n = (\sqrt{5})^n 4\sin(\theta n))$, where $\theta = \cos^{-1}\left(\frac{2}{\sqrt{5}}\right)$

4. For the following nonhomogeneous recurrence relation with initial conditions find (i) the homogeneous solution $a_n^{(h)}$, (ii) a particular solution $a_n^{(p)}$, and (iii) the solution that satisfies the initial conditions.

$$a_n - 4a_{n-1} + 3a_{n-2} = (3)2^n\,,\quad n \geq 2\,,$$
$$a_0 = 0 \quad\text{and}\quad a_1 = 0\,.$$

Ans: (i) The homogeneous solution is $a_n^{(h)} = C_1 + C_2 3^n$.

(ii) A particular solution exists in the form $a_n^{(p)} = A2^n$ for some constant A. The recurrence relation gives $A - 4A/2 + 3A/4 = 3$, so $A = -12$ and $a_n^{(p)} = -(12)2^n$.

(iii) The general solution is therefore $C_1 + C_2 3^n - (12)2^n$. The ICs, $C_1 + C_2 - 12 = 0$ and $C_1 + 3C_2 - 24 = 0$, give $C_1 = 6$ and $C_2 = 6$. Thus:

$$a_n = 6 + (6)3^n - (12)2^n$$

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5. In the ring $\mathbb{Z}_{20}$ find the following. All answers must be explicit elements of $\mathbb{Z}_{20}$. You don't have to show any work.

   (a) $3 \cdot 9 \cdot 4$ **Ans:** $(27 \bmod 20) \cdot 4 = 7 \cdot 4 = 28 \bmod 20 = 8$.

   (b) $4 \cdot (12 + 13)$ **Ans:** $4 \cdot (25 \bmod 20) = 4 \cdot 5 = 20 \bmod 20 = 0$.

   (c) -6 **Ans:** $-6 = 14$ (because $(14 + 6) \bmod 20 = 0$).

   (d) 7^{-1} (You may use trial and error.) **Ans:** $7^{-1} = 3$ (because $7 \cdot 3 = 21 \bmod 20 = 1$).

   (e) List all the proper zero divisors. **Ans:** 2, 4, 5, 6, 8, 10, 12, 14, 15, 16, 18 (elements divisible by either 2 or 5)

6. Do the following for the given rings.

   (a) The factorization of 1025 into primes is $1025 = 5^2 \cdot 41$. Find (i) the number of units and (ii) the number of proper zero divisors in the ring $\mathbb{Z}_{1025}$. You do not have to simplify, but your final answer must have no fractions.

Ans: Units: $\phi(1025) = 5^2 \cdot 41 \left(\frac{4}{5}\right)\left(\frac{40}{41}\right) = 5 \cdot 4 \cdot 40 = 800$.

Proper zero divisors: $1025 - 800 - 1 = 224$

   (b) In the ring $\mathbb{Z}_{2025}$ find $(101)^{-1}$

Ans: The Euclidean algorithm gives:

$$\begin{cases} 2025 = 20(101) + 5 \\ 101 = 20(5) + 1 \end{cases} \quad \text{or} \quad \begin{cases} n = 20k + r_1 \\ k = 20r_1 + r_2 \end{cases}$$

where $n = 2025$, $k = 101$, $r_1 = 5$, and $r_2 = 1$. Putting $r_1 = n - 20k$ (from the top equation) into the bottom one gives

$k = 20(n - 20k) + r_2 = 20n - 400k + r_2$ and solving for r_2: $r_2 = 401k - 20n$. If we put the values back in, this says $1 = 401 \cdot 101$ in the ring $\mathbb{Z}_{2025}$. Thus $(101)^{-1} = 401$.

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Second Exam B (solutions) October 27, 2023

1. For each of the following first-order recurrence relations, find the solution that satisfies the given initial condition.

   (a) $a_n = 4a_{n-1}\,,\ \ n \geq 1\,,$
   $a_0 = 11\,.$

Ans: Geometric progression: $a_n = (11)4^n$

   (b) $a_n = a_{n-1} + 5\,,\ \ n \geq 1\,,$
   $a_0 = 8\,.$

Ans: Arithmetic progression: $a_n = 8 + 5n$

   (c) $a_n = (n^2 + 5)a_{n-1}\,,\ \ n \geq 1\,,$
   $a_0 = 5\,.$

Ans: Successive multiplications: $a_n = 5(1^2 + 5)(2^2 + 5) \cdots (n^2 + 5)$.

   (d) $a_n = a_{n-1} + n^3 5^n\,,\ \ n \geq 1\,,$
   $a_0 = 2\,.$

Ans: Successive additions: $a_n = 2 + (1^3)5^1 + (2^3)5^2 + \cdots + n^3 5^n$

2. For the following second-order recurrence relation, find (i) the characteristic equation and its roots, (ii) the general solution, and (iii) the solution that satisfies the initial conditions.

$$a_n - 6a_{n-1} + 9a_{n-2} = 0\,,\ \ n \geq 2\,,$$
$$a_0 = 1 \ \text{ and } \ a_1 = 5,.$$

Ans: (i) $r^2 - 6r + 9 = 0$ has roots 3 and 3 (repeated root).

(ii) General solution: $C_1 3^n + C_2 n 3^n$.

(iii) The ICs, $C_1 = 1$ and $3C_1 + 3C_2 = 5$, give $C_1 = 1$ and $C_2 = 2/3$. Thus:

$$a_n = 3^n + \frac{2}{3} n 3^n$$

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3. For the following second-order recurrence relation, find (i) the characteristic equation and its roots, (ii) the general solution, and (iii) the solution that satisfies the initial conditions.

$$a_n - 2a_{n-1} + 10a_{n-2} = 0\,, \quad n \geq 2\,,$$
$$a_0 = 0 \quad \text{and} \quad a_1 = 6.$$

Ans: (i) $r^2 - 2r + 10 = 0$ has roots $r = 1 \pm 3i$.

(ii) General solution: $C_1(1+3i)^n + C_2(1-3i)^n$.

(iii) The ICs, $C_1 + C_2 = 0$ and $C_1(1+3i) + C_2(1-3i) = 6$, give $C_1 = -i$ and $C_2 = i$.

Thus: $a_n = -i(1+3i)^n + i(1-3i)^n$.

Or use the form $a_n = (\sqrt{10})^n 2\sin(\theta n))$, where $\theta = \cos^{-1}\left(\frac{1}{\sqrt{10}}\right)$

4. For the following nonhomogeneous recurrence relation with initial conditions find (i) the homogeneous solution $a_n^{(h)}$, (ii) a particular solution $a_n^{(p)}$, and (iii) the solution that satisfies the initial conditions.

$$a_n - 5a_{n-1} + 4a_{n-2} = (3)2^n\,, \quad n \geq 2\,,$$
$$a_0 = 0 \quad \text{and} \quad a_1 = 0\,.$$

Ans: (i) The homogeneous solution is $a_n^{(h)} = C_1 + C_2 4^n$.

(ii) A particular solution exists in the form $a_n^{(p)} = A2^n$ for some constant A. The recurrence relation gives $A - 5A/2 + 4A/4 = 3$, so $A = -6$ and $a_n^{(p)} = -(6)2^n$.

(iii) The general solution is therefore $C_1 + C_2 4^n - (6)2^n$. The ICs, $C_1 + C_2 - 6 = 0$ and $C_1 + 4C_2 - 12 = 0$, give $C_1 = 4$ and $C_2 = 2$. Thus:

$$a_n = 4 + (2)4^n - (6)2^n$$

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5. In the ring $\mathbb{Z}_{21}$ find the following. All answers must be explicit elements of $\mathbb{Z}_{21}$. You don't have to show any work.

(a) $3 \cdot 9 \cdot 4$ **Ans:** $(27 \bmod 21) \cdot 4 = 6 \cdot 4 = 24 \bmod 21 = 3$.

(b) $4 \cdot (12 + 13)$ **Ans:** $4 \cdot (25 \bmod 21) = 4 \cdot 4 = 16 \bmod 21 = 16$.

(c) -6 **Ans:** $-6 = 15$ (because $(15 + 6) \bmod 21 = 0$).

(d) 11^{-1} (You may use trial and error.) **Ans:** $11^{-1} = 2$ (because $11 \cdot 2 = 22 \bmod 21 = 1$).

(e) List all the proper zero divisors. **Ans:** $3, 6, 9, 12, 15, 18, 7, 14$ (elements divisible by either 3 or 7)

6. Do the following for the given rings.

(a) The factorization of 1026 into primes is $1026 = 2 \cdot 3^3 \cdot 19$. Find (i) the number of units and (ii) the number of proper zero divisors in the ring $\mathbb{Z}_{1026}$. You do not have to simplify, but your final answer must have no fractions.

Ans: Units: $\phi(1026) = 2 \cdot 3^3 \cdot 19 \left(\frac{1}{2}\right)\left(\frac{2}{3}\right)\left(\frac{18}{19}\right) = 3^2 \cdot 2 \cdot 18 = 324$.

Proper zero divisors: $1026 - 324 - 1 = 701$

(b) In the ring $\mathbb{Z}_{2011}$ find $(100)^{-1}$

Ans: The Euclidean algorithm gives:

$$\begin{cases} 2011 = 20(100) + 11 \\ 100 = 9(11) + 1 \end{cases} \quad \text{or} \quad \begin{cases} n = 20k + r_1 \\ k = 9r_1 + r_2 \end{cases}$$

where $n = 2011$, $k = 100$, $r_1 = 11$, and $r_2 = 1$. Putting $r_1 = n - 20k$ (from the top equation) into the bottom one gives

$k = 9(n - 20k) + r_2 = 9n - 180k + r_2$ and solving for r_2: $r_2 = 181k - 9n$. If we put the values back in, this says $1 = 181 \cdot 100$ in the ring $\mathbb{Z}_{2011}$. Thus $(100)^{-1} = 181$.

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Second Exam C (solutions) October 27, 2023

1. For each of the following first-order recurrence relations, find the solution that satisfies the given initial condition.

   (a) $a_n = 5a_{n-1}\,,\ \ n \geq 1\,,$
   $a_0 = 12\,.$

Ans: Geometric progression: $a_n = (12)5^n$

   (b) $a_n = a_{n-1} + 6\,,\ \ n \geq 1\,,$
   $a_0 = 9\,.$

Ans: Arithmetic progression: $a_n = 9 + 6n$

   (c) $a_n = (n^2 + 6)a_{n-1}\,,\ \ n \geq 1\,,$
   $a_0 = 5\,.$

Ans: Successive multiplications: $a_n = 5(1^2 + 6)(2^2 + 6)\cdots(n^2 + 6)$.

   (d) $a_n = a_{n-1} + n^3 6^n\,,\ \ n \geq 1\,,$
   $a_0 = 2\,.$

Ans: Successive additions: $a_n = 2 + (1^3)6^1 + (2^3)6^2 + \cdots + n^3 6^n$

2. For the following second-order recurrence relation, find (i) the characteristic equation and its roots, (ii) the general solution, and (iii) the solution that satisfies the initial conditions.

   $a_n - 6a_{n-1} + 9a_{n-2} = 0\,,\ \ n \geq 2\,,$
   $a_0 = 1$ and $a_1 = 7,.$

Ans: (i) $r^2 - 6r + 9 = 0$ has roots 3 and 3 (repeated root).

(ii) General solution: $C_1 3^n + C_2 n 3^n$.

(iii) The ICs, $C_1 = 1$ and $3C_1 + 3C_2 = 7$, give $C_1 = 1$ and $C_2 = 4/3$. Thus:

$$a_n = 3^n + \frac{4}{3} n 3^n$$

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3. For the following second-order recurrence relation, find (i) the characteristic equation and its roots, (ii) the general solution, and (iii) the solution that satisfies the initial conditions.

$$a_n - 4a_{n-1} + 5a_{n-2} = 0\,, \quad n \geq 2\,,$$
$$a_0 = 0 \quad \text{and} \quad a_1 = 6.$$

Ans: (i) $r^2 - 4r + 5 = 0$ has roots $r = 2 \pm i$.

(ii) General solution: $C_1(2+i)^n + C_2(2-i)^n$.

(iii) The ICs, $C_1 + C_2 = 0$ and $C_1(2+i) + C_2(2-i) = 6$, give $C_1 = -3i$ and $C_2 = 3i$.

Thus: $a_n = -3i(2+i)^n + 3i(2-i)^n$.

Or use the form $a_n = (\sqrt{5})^n 6\sin(\theta n))$, where $\theta = \cos^{-1}\left(\frac{2}{\sqrt{5}}\right)$

4. For the following nonhomogeneous recurrence relation with initial conditions find (i) the homogeneous solution $a_n^{(h)}$, (ii) a particular solution $a_n^{(p)}$, and (iii) the solution that satisfies the initial conditions.

$$a_n - 6a_{n-1} + 5a_{n-2} = (6)2^n\,, \quad n \geq 2\,,$$
$$a_0 = 0 \quad \text{and} \quad a_1 = 0\,.$$

Ans: (i) The homogeneous solution is $a_n^{(h)} = C_1 + C_2 5^n$.

(ii) A particular solution exists in the form $a_n^{(p)} = A2^n$ for some constant A. The recurrence relation gives $A - 6A/2 + 5A/4 = 6$, so $A = -8$ and $a_n^{(p)} = -(8)2^n$.

(iii) The general solution is therefore $C_1 + C_2 5^n - (8)2^n$. The ICs, $C_1 + C_2 - 8 = 0$ and $C_1 + 5C_2 - 16 = 0$, give $C_1 = 6$ and $C_2 = 2$. Thus:

$$a_n = 6 + (2)5^n - (8)2^n$$

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5. In the ring $\mathbb{Z}_{22}$ find the following. All answers must be explicit elements of $\mathbb{Z}_{22}$. You don't have to show any work.

(a) $3 \cdot 9 \cdot 4$ **Ans:** $(27 \bmod 22) \cdot 4 = 5 \cdot 4 = 20 \bmod 22 = 20$.

(b) $4 \cdot (12 + 13)$ **Ans:** $4 \cdot (25 \bmod 22) = 4 \cdot 3 = 12 \bmod 22 = 12$.

(c) -6 **Ans:** $-6 = 16$ (because $(16 + 6) \bmod 22 = 0$).

(d) 15^{-1} (You may use trial and error.) **Ans:** $15^{-1} = 3$ (because $15 \cdot 3 = 45 \bmod 22 = 1$).

(e) List all the proper zero divisors. **Ans:** $2, 4, 6, 8, 10, 12, 14, 16, 18, 20, 11$ (elements divisible by either 2 or 11)

6. Do the following for the given rings.

(a) The factorization of 1028 into primes is $1028 = 2^2 \cdot 257$. Find (i) the number of units and (ii) the number of proper zero divisors in the ring $\mathbb{Z}_{1028}$. You do not have to simplify, but your final answer must have no fractions.

Ans: Units: $\phi(1028) = 2^2 \cdot 257 \left(\frac{1}{2}\right)\left(\frac{256}{257}\right) = 2 \cdot 256 = 512$.

Proper zero divisors: $1028 - 512 - 1 = 515$

(b) In the ring $\mathbb{Z}_{2030}$ find $(101)^{-1}$

Ans: The Euclidean algorithm gives:

$$\begin{cases} 2030 = 20(101) + 10 \\ 101 = 10(10) + 1 \end{cases} \quad \text{or} \quad \begin{cases} n = 20k \ + r_1 \\ k = 10r_1 + r_2 \end{cases}$$

where $n = 2030$, $k = 101$, $r_1 = 10$, and $r_2 = 1$. Putting $r_1 = n - 20k$ (from the top equation) into the bottom one gives
$k = 10(n - 20k) + r_2 = 10n - 200k + r_2$ and solving for r_2: $r_2 = 201k - 10n$. If we put the values back in, this says $1 = 201 \cdot 101$ in the ring $\mathbb{Z}_{2030}$. Thus $(101)^{-1} = 201$.

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