The Pattern Inventory

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MASC

November 29, 2023

Group element	cycle structure representation	
$(1)(2)(3)(4)$	$x_1x_1x_1x_1$	$= x_1^4$
$(13)(24)$	x_2x_2	$= x_2^2$
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For our current example:

$$\begin{aligned} P_G(r+w, r^2+w^2) &= \frac{(r+w)^4 + (r^2+w^2)^2 + 2(r+w)^2(r^2+w^2)}{4} \\ &= r^4 + 2r^3w + 3r^2w^2 + 2rw^3 + w^4, \end{aligned}$$

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This is called the pattern inventory.

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$$P_G(r+w, r^2+w^2) = r^4 + 2r^3w + 3r^2w^2 + 2rw^3 + w^4,$$

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If there are three colors such as red, white and blue, we would substitute $x_1 = r + w + b$, $x_2 = r^2 + w^2 + b^2$, etc.

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[However, if we build it with balls at the 4 vertices and wires connecting them that can bend and twist a bit, then all permutations are possible.]

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And the number of distinguishable colorings of the vertices with 2 colors is $(2^4 + 8 \cdot 2^2 + 3 \cdot 2^2)/12 = 5$. With 3 colors it is $(3^4 + 8 \cdot 3^2 + 3 \cdot 3^2)/12 = 15$.

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And so the number of distinguishable colorings of the sides with two colors is $(2^4 + 2^2 + 2^3 + 2^3)/4 = 9$. Whereas for colorings of the vertices it was 7.