Groups and Burnside's Theorem

Daniel H. Luecking

MASC

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If we use more details of the cycles, we can answer questions such as: How many distinguishable colorings are there in which 2 vertices are blue.

For this we also need to keep track of the length of each cycle. We keep track of this with another sort of generating function.

Let's work on the group for the equilateral triangle. For each element of the group let's substitute the variable x_1 for each 1-cycle (cycle of length 1), x_2 for each 2-cycle (cycle of length 2), x_3 for each 3-cycle, etc.:

$$G : \quad (1)(2)(3) \quad (123) \quad (132) \quad (1)(23) \quad (2)(13) \quad (3)(12)$$
$$\qquad\quad x_1 x_1 x_1 \qquad x_3 \qquad x_3 \qquad x_1 x_2 \qquad x_1 x_2 \qquad x_1 x_2$$

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Then we add these all up, collect like terms and divide by the order of the group:

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On the next slide we do this for the rectangle's group.

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Here are the computations for the group of the square: $G = \{(1)(2)(3)(4),$
$(1234), (13)(24), (1432), (12)(34), (14)(23), (1)(24)(3), (2)(13)(4)\}$

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$$P_G(2, 2, 2, 2) = \frac{1}{4}(2^4 + 3 \cdot 2^2) = \frac{28}{4} = 7.$$

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$$P_G(r + w, r^2 + w^2, r^3 + w^3, r^4 + w^4) = \frac{1}{4}[(r+w)^4 + 3(r^2 + w^2)^2]$$
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The number in front of each term tells us how many distinguishable colorings have that combination of colors: there is only one that uses 4 red, or 3 red and 1 white, or 3 white and 1 red, or 4 white.

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The number in front of each term tells us how many distinguishable colorings have that combination of colors: there is only one that uses 4 red, or 3 red and 1 white, or 3 white and 1 red, or 4 white. But there are 3 distinguishable colorings that use 2 red and 2 white.

Let's go over again the process for creating the *cycle index polynomial* for the group of rigid motions of a figure.
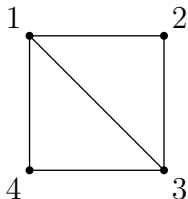
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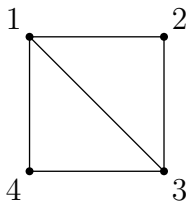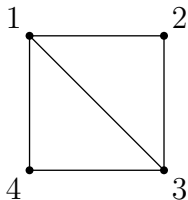
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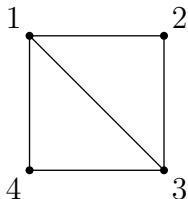
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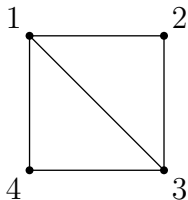
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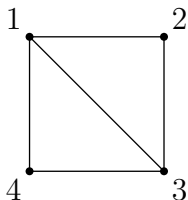

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For our current example:

$$\begin{aligned}
P_G(r + w, r^2 + w^2) &= \frac{(r+w)^4 + (r^2+w^2)^2 + 2(r+w)^2(r^2+w^2)}{4} \\
&= r^4 + 2r^3w + 3r^2w^2 + 2rw^3 + w^4,
\end{aligned}$$

The term $3r^2w^2$ tells us there are 3 distinguishable colorings in which 2 vertices are red and 2 are white.
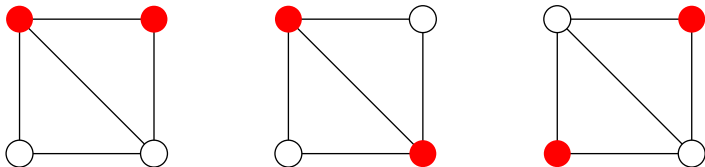
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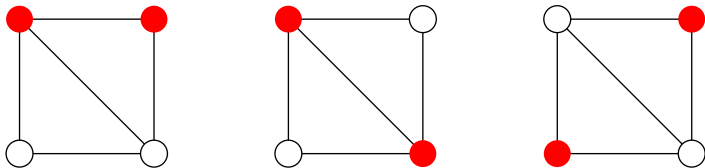
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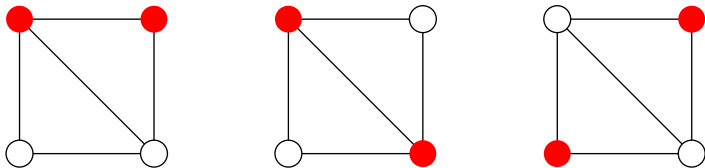
Here are the three that use 2 red and 2 white

If there are three colors such as red, white and blue, we would substitute $x_1 = r + w + b$, $x_2 = r^2 + w^2 + b^2$, etc.

The term $3r^2w^2$ tells us there are 3 distinguishable colorings in which 2 vertices are red and 2 are white. And the term $2rw^3$ says there are 2 distinguishable colorings with 1 red and 3 white vertices. It is difficult to explain why this works. I refer you to the textbook if you are sufficiently curious.
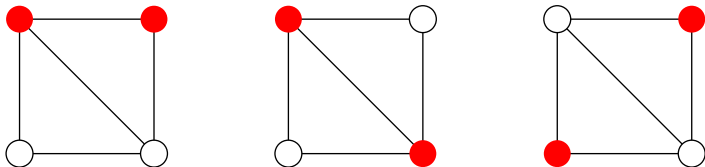
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Without showing any details, for our current group:

$$P_G(r+b+w, r^2+b^2+w^2) = r^4 + w^4 + b^4 + 2r^3w + 2r^3b + 2w^3r + 2w^3b$$
$$+ 2b^3r + 2b^3w + 3r^2w^2 + 3r^2b^2 + 3w^2b^2 + 4rwb^2 + 4rbw^2 + 4wbr^2$$

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(I did this by hand . . . I don't recommend doing that.)

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Without showing any details, for our current group:

$$P_G(r+b+w, r^2+b^2+w^2) = r^4 + w^4 + b^4 + 2r^3w + 2r^3b + 2w^3r + 2w^3b$$
$$+ 2b^3r + 2b^3w + 3r^2w^2 + 3r^2b^2 + 3w^2b^2 + 4rwb^2 + 4rbw^2 + 4wbr^2$$

(I did this by hand … I don't recommend doing that.) We can see from the last term (for example) there are 4 distinguishable ways to color the vertices where 2 are red, one white and one blue. Here they are: