Groups and Burnside's Theorem

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MASC

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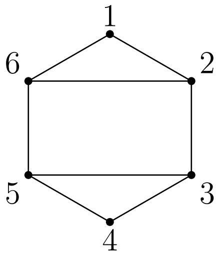
More examples of groups of rigid motions



A regular hexagon with two added lines

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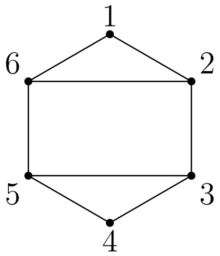

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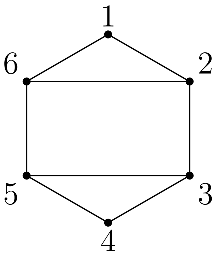

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The identity: (1)(2)(3)(4)(5)(6).

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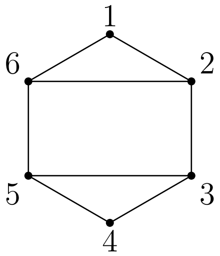

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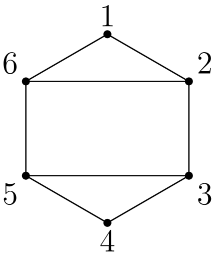

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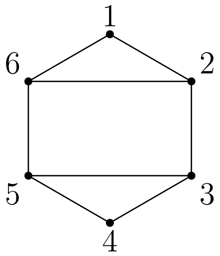

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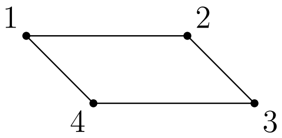
A parallelogram:



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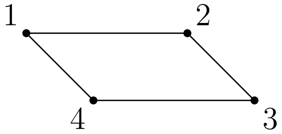

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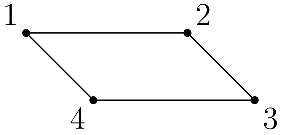

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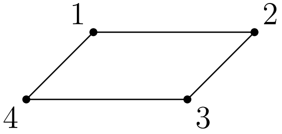
Not the same parallelogram:



But the same rigid motions: (1)(2)(3)(4) and (13)(24).

Here's a brief explanation of Burnside's Theorem for the group of rigid motions of an equilateral triangle.

$\mathscr{C} \backslash G$	(1)(2)(3)	(123)	(132)	(1)(23)	(2)(13)	(3)(12)		
WWW	1	1	1	1	1	1	6	6
RWW	1	0	0	1	0	0	2	
WRW	1	0	0	0	1	0	2	
WWR	1	0	0	0	0	1	2	6
WRR	1	0	0	1	0	0	2	
RWR	1	0	0	0	1	0	2	
RRW	1	0	0	0	0	1	2	6
RRR	1	1	1	1	1	1	6	6
	8	2	2	4	4	4		24

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So, for our equilateral triangle we have the following table, where all we have to do is take the number of colors to the power of the number of cycles.

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$(1)(2)(3)$	(123)	(132)	$(1)(23)$	$(2)(13)$	$(3)(12)$
2^3	2^1	2^1	2^2	2^2	2^2

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The group of rigid motions of the square is

$$\{(1)(2)(3)(4),(1234),(13)(24),(1432),(12)(34),(14)(23),(1)(24)(3),(13)(2)(4)\}$$

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With 2 colors, the number of distinguishable colorings is

$$\frac{1}{8}\left(2^4+2^1+2^2+2^1+2^2+2^2+2^3+2^3\right)=\frac{48}{8}=6.$$

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The group of rigid motions of the rectangle is

$$G = \{(1)(2)(3)(4), (13)(24), (12)(34), (14)(23)\}$$

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For the equilateral triangle, with three colors, the number of distinguishable colorings is

$$\frac{1}{6}\left(3^3 + 3^1 + 3^1 + 3^2 + 3^2 + 3^2\right) = \frac{60}{6} = 10.$$

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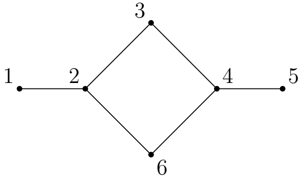

A square with two symmetrical lines added.

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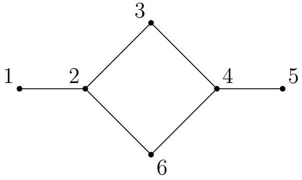

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The group of the above figure is
$\{(1)(2)(3)(4)(5)(6), (15)(24)(36), (15)(24)(3)(6), (1)(2)(36)(4)(5)\}$.

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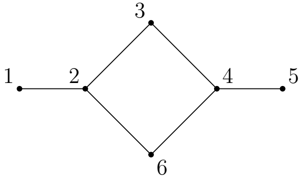

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The group of the above figure is $\{(1)(2)(3)(4)(5)(6), (15)(24)(36), (15)(24)(3)(6), (1)(2)(36)(4)(5)\}$. The number of distinguishable colorings with 5 colors:

$$\frac{1}{4}\left(5^6 + 5^3 + 5^4 + 5^5\right) = 4875$$

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$$\text{2 colors:} \quad \frac{1}{2}\left(2^4 + 2^2\right) = 10$$

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$$\text{4 colors:} \quad \frac{1}{2}\left(4^4 + 4^2\right) = 136$$

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One last example: the regular pentagon, whose group is

$$\{(1)(2)(3)(4)(5), (12345), (13524), (14253), (15432), (1)(25)(34), \\ (2)(13)(54), (3)(24)(15), (4)(35)(12), (5)(14)(23)\}.$$

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Exercise: for 2 colors, compute the number of distinguishable colorings of the vertices of a regular hexagon.