Groups and Symmetry

Daniel H. Luecking
MASC

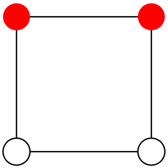
November 15, 2023

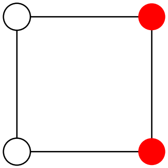
Symmetries

To illustrate the use of symmetry in combinatorics, we pose the following problem: We build a geometric figure out of identical sticks (except possibly different lengths) and identical disks (except that the disks come in two or more colors).

Symmetries

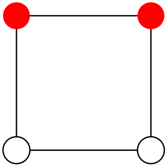
To illustrate the use of symmetry in combinatorics, we pose the following problem: We build a geometric figure out of identical sticks (except possibly different lengths) and identical disks (except that the disks come in two or more colors). Below we illustrate with two squares with disks colored red and white.

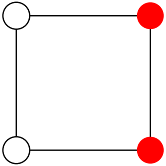




Symmetries

To illustrate the use of symmetry in combinatorics, we pose the following problem: We build a geometric figure out of identical sticks (except possibly different lengths) and identical disks (except that the disks come in two or more colors). Below we illustrate with two squares with disks colored red and white.

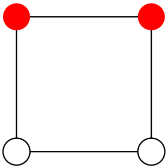


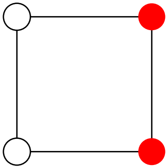


Note that the two squares are not necessarily different. The second could just be the first one rotated 90° clockwise.

Symmetries

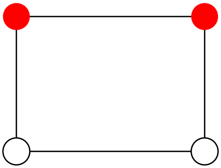
To illustrate the use of symmetry in combinatorics, we pose the following problem: We build a geometric figure out of identical sticks (except possibly different lengths) and identical disks (except that the disks come in two or more colors). Below we illustrate with two squares with disks colored red and white.

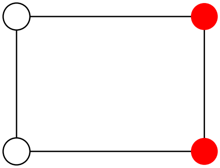




Note that the two squares are not necessarily different. The second could just be the first one rotated 90° clockwise. We say these ways of coloring the vertices are *indistinguishable*.

Now consider the following:





Now consider the following:

These two rectangles are definitely different, because one has red disks on a long side and the other has red disks on a short side and no amount of moving will change that.

Now consider the following:

These two rectangles are definitely different, because one has red disks on a long side and the other has red disks on a short side and no amount of moving will change that.

The combinatorial question is: given that we are allowed to move the figures, how many distinguishable colorings are there?

Now consider the following:

These two rectangles are definitely different, because one has red disks on a long side and the other has red disks on a short side and no amount of moving will change that.

The combinatorial question is: given that we are allowed to move the figures, how many distinguishable colorings are there? The book illustrates this for the square by exhibiting all possible colorings of a stationary square, and grouping them by which can be turned into each other by moving the square.

Now consider the following:

These two rectangles are definitely different, because one has red disks on a long side and the other has red disks on a short side and no amount of moving will change that.

The combinatorial question is: given that we are allowed to move the figures, how many distinguishable colorings are there? The book illustrates this for the square by exhibiting all possible colorings of a stationary square, and grouping them by which can be turned into each other by moving the square. There are $2^4 = 16$ figures and 6 groups. The answer is 6 distinguishable colorings.

The previous answer assumed any rigid motion of the square is allowed.

Suppose that no motion of the square is possible (say we glue the figure to a wall). Then all 16 possibilities are distinguishable.

The previous answer assumed any rigid motion of the square is allowed.

Suppose that no motion of the square is possible (say we glue the figure to a wall). Then all 16 possibilities are distinguishable. So, the number of distinguishable colorings depends on what motions are allowed or are possible.

The previous answer assumed any rigid motion of the square is allowed.

Suppose that no motion of the square is possible (say we glue the figure to a wall). Then all 16 possibilities are distinguishable. So, the number of distinguishable colorings depends on what motions are allowed or are possible. This makes a difference between the rectangle and the square. A rotation by 90° turns the square into itself.

The previous answer assumed any rigid motion of the square is allowed.

Suppose that no motion of the square is possible (say we glue the figure to a wall). Then all 16 possibilities are distinguishable. So, the number of distinguishable colorings depends on what motions are allowed or are possible. This makes a difference between the rectangle and the square. A rotation by 90° turns the square into itself. That is, if the vertices have not been colored, you can't even tell the square has moved.

The previous answer assumed any rigid motion of the square is allowed.

Suppose that no motion of the square is possible (say we glue the figure to a wall). Then all 16 possibilities are distinguishable. So, the number of distinguishable colorings depends on what motions are allowed or are possible. This makes a difference between the rectangle and the square. A rotation by 90° turns the square into itself. That is, if the vertices have not been colored, you can't even tell the square has moved. However, rotating a rectangle 90° is definitely noticeable.

The previous answer assumed any rigid motion of the square is allowed.

Suppose that no motion of the square is possible (say we glue the figure to a wall). Then all 16 possibilities are distinguishable. So, the number of distinguishable colorings depends on what motions are allowed or are possible. This makes a difference between the rectangle and the square. A rotation by 90° turns the square into itself. That is, if the vertices have not been colored, you can't even tell the square has moved. However, rotating a rectangle 90° is definitely noticeable.

It turns out there are 7 distinguishable ways to color the rectangle, versus only 6 for the square and the difference lies in the structure of the *group of rigid motions*.

The previous answer assumed any rigid motion of the square is allowed.

Suppose that no motion of the square is possible (say we glue the figure to a wall). Then all 16 possibilities are distinguishable. So, the number of distinguishable colorings depends on what motions are allowed or are possible. This makes a difference between the rectangle and the square. A rotation by 90° turns the square into itself. That is, if the vertices have not been colored, you can't even tell the square has moved. However, rotating a rectangle 90° is definitely noticeable.

It turns out there are 7 distinguishable ways to color the rectangle, versus only 6 for the square and the difference lies in the structure of the *group of rigid motions*.

A rigid motion of a geometric figure (also called a *congruence*) is a motion of the figure that leaves it unchanged.

The previous answer assumed any rigid motion of the square is allowed.

Suppose that no motion of the square is possible (say we glue the figure to a wall). Then all 16 possibilities are distinguishable. So, the number of distinguishable colorings depends on what motions are allowed or are possible. This makes a difference between the rectangle and the square. A rotation by 90° turns the square into itself. That is, if the vertices have not been colored, you can't even tell the square has moved. However, rotating a rectangle 90° is definitely noticeable.

It turns out there are 7 distinguishable ways to color the rectangle, versus only 6 for the square and the difference lies in the structure of the *group of rigid motions*.

A rigid motion of a geometric figure (also called a *congruence*) is a motion of the figure that leaves it unchanged. That is, after the motion, the figure appears unchanged. What motions are possible is determined by the symmetry of the figure.

The previous answer assumed any rigid motion of the square is allowed.

Suppose that no motion of the square is possible (say we glue the figure to a wall). Then all 16 possibilities are distinguishable. So, the number of distinguishable colorings depends on what motions are allowed or are possible. This makes a difference between the rectangle and the square. A rotation by 90° turns the square into itself. That is, if the vertices have not been colored, you can't even tell the square has moved. However, rotating a rectangle 90° is definitely noticeable.

It turns out there are 7 distinguishable ways to color the rectangle, versus only 6 for the square and the difference lies in the structure of the *group of rigid motions*.

A rigid motion of a geometric figure (also called a *congruence*) is a motion of the figure that leaves it unchanged. That is, after the motion, the figure appears unchanged. What motions are possible is determined by the symmetry of the figure. The book works out all the rigid motions of the square. I'll do the same for the rectangle.

All symmetries of 2-D figures are one of two types: there are lines of symmetry and rotation symmetry.

All symmetries of 2-D figures are one of two types: there are lines of symmetry and rotation symmetry. Any line of symmetry is associated with a motion (called *reflection*) that flips the figure over with that line as an axis.

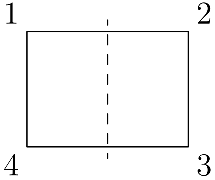
All symmetries of 2-D figures are one of two types: there are lines of symmetry and rotation symmetry. Any line of symmetry is associated with a motion (called *reflection*) that flips the figure over with that line as an axis. A rotation symmetry gives rise to a motion that rotates the figure by some angle, leaving the figure unchanged.

All symmetries of 2-D figures are one of two types: there are lines of symmetry and rotation symmetry. Any line of symmetry is associated with a motion (called *reflection*) that flips the figure over with that line as an axis. A rotation symmetry gives rise to a motion that rotates the figure by some angle, leaving the figure unchanged.

For the rectangle we have 2 reflection symmetries.

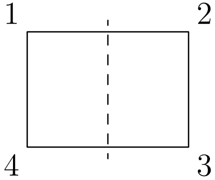
All symmetries of 2-D figures are one of two types: there are lines of symmetry and rotation symmetry. Any line of symmetry is associated with a motion (called *reflection*) that flips the figure over with that line as an axis. A rotation symmetry gives rise to a motion that rotates the figure by some angle, leaving the figure unchanged.

For the rectangle we have 2 reflection symmetries. One of these is a vertical line of symmetry through the midpoints of the top and bottom sides:



All symmetries of 2-D figures are one of two types: there are lines of symmetry and rotation symmetry. Any line of symmetry is associated with a motion (called *reflection*) that flips the figure over with that line as an axis. A rotation symmetry gives rise to a motion that rotates the figure by some angle, leaving the figure unchanged.

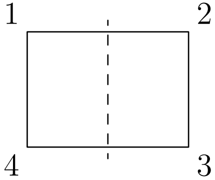
For the rectangle we have 2 reflection symmetries. One of these is a vertical line of symmetry through the midpoints of the top and bottom sides:



The reflection associated with this line exchanges the two vertices labeled 1 and 2, as well as 3 and 4.

All symmetries of 2-D figures are one of two types: there are lines of symmetry and rotation symmetry. Any line of symmetry is associated with a motion (called *reflection*) that flips the figure over with that line as an axis. A rotation symmetry gives rise to a motion that rotates the figure by some angle, leaving the figure unchanged.

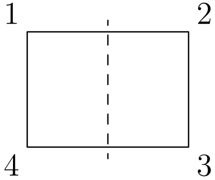
For the rectangle we have 2 reflection symmetries. One of these is a vertical line of symmetry through the midpoints of the top and bottom sides:



The reflection associated with this line exchanges the two vertices labeled 1 and 2, as well as 3 and 4. There is a horizontal line of symmetry and its reflection exchanges 1 and 4, as well as 2 and 3.

All symmetries of 2-D figures are one of two types: there are lines of symmetry and rotation symmetry. Any line of symmetry is associated with a motion (called *reflection*) that flips the figure over with that line as an axis. A rotation symmetry gives rise to a motion that rotates the figure by some angle, leaving the figure unchanged.

For the rectangle we have 2 reflection symmetries. One of these is a vertical line of symmetry through the midpoints of the top and bottom sides:



The reflection associated with this line exchanges the two vertices labeled 1 and 2, as well as 3 and 4. There is a horizontal line of symmetry and its reflection exchanges 1 and 4, as well as 2 and 3. A 180° rotation will exchange 1 and 3 as well as 2 and 4.

One final rigid motion of the rectangle is the identity (i.e., no motion at all).

One final rigid motion of the rectangle is the identity (i.e., no motion at all). This is needed to have a group and is usually thought of as a rotation by 0° (or sometimes 360°).

One final rigid motion of the rectangle is the identity (i.e., no motion at all). This is needed to have a group and is usually thought of as a rotation by 0° (or sometimes 360°). This is in fact the entire group (and it is a group).

One final rigid motion of the rectangle is the identity (i.e., no motion at all). This is needed to have a group and is usually thought of as a rotation by 0° (or sometimes 360°). This is in fact the entire group (and it is a group). We'll use π_0 for the identity, π_1 for 180° rotation, r_1 for the top-to-bottom reflection and r_2 for the left-to-right reflection.

One final rigid motion of the rectangle is the identity (i.e., no motion at all). This is needed to have a group and is usually thought of as a rotation by 0° (or sometimes 360°). This is in fact the entire group (and it is a group). We'll use π_0 for the identity, π_1 for 180° rotation, r_1 for the top-to-bottom reflection and r_2 for the left-to-right reflection. (The square has two more rotations: 90° and 270° and two diagonal lines of symmetry producing 2 more reflections.)

One final rigid motion of the rectangle is the identity (i.e., no motion at all). This is needed to have a group and is usually thought of as a rotation by 0° (or sometimes 360°). This is in fact the entire group (and it is a group). We'll use π_0 for the identity, π_1 for 180° rotation, r_1 for the top-to-bottom reflection and r_2 for the left-to-right reflection. (The square has two more rotations: 90° and 270° and two diagonal lines of symmetry producing 2 more reflections.)

In the group of rigid motions, the group operation is simple: apply the two motions in succession.

One final rigid motion of the rectangle is the identity (i.e., no motion at all). This is needed to have a group and is usually thought of as a rotation by 0° (or sometimes 360°). This is in fact the entire group (and it is a group). We'll use π_0 for the identity, π_1 for 180° rotation, r_1 for the top-to-bottom reflection and r_2 for the left-to-right reflection. (The square has two more rotations: 90° and 270° and two diagonal lines of symmetry producing 2 more reflections.)

In the group of rigid motions, the group operation is simple: apply the two motions in succession. For the rectangle group $\pi_1\pi_1$ is two 180° rotations in a row, putting every point back where it was, that is $\pi_1\pi_1 = \pi_0$.

One final rigid motion of the rectangle is the identity (i.e., no motion at all). This is needed to have a group and is usually thought of as a rotation by 0° (or sometimes 360°). This is in fact the entire group (and it is a group). We'll use π_0 for the identity, π_1 for 180° rotation, r_1 for the top-to-bottom reflection and r_2 for the left-to-right reflection. (The square has two more rotations: 90° and 270° and two diagonal lines of symmetry producing 2 more reflections.)

In the group of rigid motions, the group operation is simple: apply the two motions in succession. For the rectangle group $\pi_1\pi_1$ is two 180° rotations in a row, putting every point back where it was, that is $\pi_1\pi_1 = \pi_0$. In fact, every element of $G = \{\pi_0, \pi_1, r_1, r_2\}$ is its own inverse. G is closed under this operation. For example $\pi_1 r_1 = r_2$.

One final rigid motion of the rectangle is the identity (i.e., no motion at all). This is needed to have a group and is usually thought of as a rotation by 0° (or sometimes 360°). This is in fact the entire group (and it is a group). We'll use π_0 for the identity, π_1 for 180° rotation, r_1 for the top-to-bottom reflection and r_2 for the left-to-right reflection. (The square has two more rotations: 90° and 270° and two diagonal lines of symmetry producing 2 more reflections.)

In the group of rigid motions, the group operation is simple: apply the two motions in succession. For the rectangle group $\pi_1\pi_1$ is two 180° rotations in a row, putting every point back where it was, that is $\pi_1\pi_1 = \pi_0$. In fact, every element of $G = \{\pi_0, \pi_1, r_1, r_2\}$ is its own inverse. G is closed under this operation. For example $\pi_1 r_1 = r_2$. This can be verified by working out where each motion puts each vertex.

In fact, each motion is completely determined by what it does to the vertices. Because the motions are rigid, the lines connecting vertices move along with the vertices.

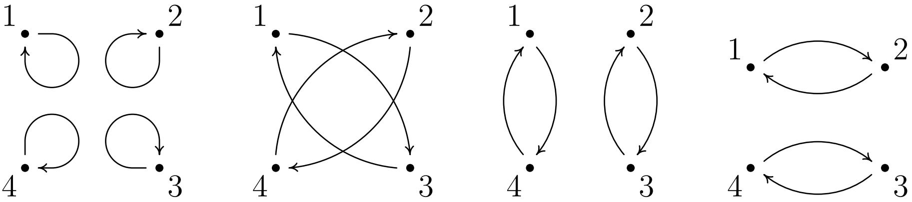
In fact, each motion is completely determined by what it does to the vertices. Because the motions are rigid, the lines connecting vertices move along with the vertices. Thus each motion can be represented as a permutation of the vertices. For the rectangle group we have

$$G = \left\{ \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 4 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 4 & 1 & 2 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 3 & 2 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \end{pmatrix} \right\}$$

In fact, each motion is completely determined by what it does to the vertices. Because the motions are rigid, the lines connecting vertices move along with the vertices. Thus each motion can be represented as a permutation of the vertices. For the rectangle group we have

$$G = \left\{ \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 4 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 4 & 1 & 2 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 3 & 2 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \end{pmatrix} \right\}$$

We can visualize these permutations caused by motions using points and arrows:



The stacked notation for permutations is awkward for what we are about to do.

The stacked notation for permutations is awkward for what we are about to do. The figures-with-arrows is useful, but takes up too much space.

The stacked notation for permutations is awkward for what we are about to do. The figures-with-arrows is useful, but takes up too much space. So we introduce a compromise called the *disjoint cycle notation*.

The stacked notation for permutations is awkward for what we are about to do. The figures-with-arrows is useful, but takes up too much space. So we introduce a compromise called the *disjoint cycle notation*.

In this we take each closed cycle in the points-with-arrows figure and list it (in order of the arrows) in parentheses.

The stacked notation for permutations is awkward for what we are about to do. The figures-with-arrows is useful, but takes up too much space. So we introduce a compromise called the *disjoint cycle notation*.

In this we take each closed cycle in the points-with-arrows figure and list it (in order of the arrows) in parentheses. Thus our entire group G can be written

$$G = \{(1)(2)(3)(4), (13)(24), (14)(23), (12)(34)\}$$

The stacked notation for permutations is awkward for what we are about to do. The figures-with-arrows is useful, but takes up too much space. So we introduce a compromise called the *disjoint cycle notation*.

In this we take each closed cycle in the points-with-arrows figure and list it (in order of the arrows) in parentheses. Thus our entire group G can be written

$$G = \{(1)(2)(3)(4), (13)(24), (14)(23), (12)(34)\}$$

A sequence of vertex labels in parentheses like $(abc \ldots r)$ is saying that the motion it represents moves a to b, b to c,..., and r to a.

The stacked notation for permutations is awkward for what we are about to do. The figures-with-arrows is useful, but takes up too much space. So we introduce a compromise called the *disjoint cycle notation*.

In this we take each closed cycle in the points-with-arrows figure and list it (in order of the arrows) in parentheses. Thus our entire group G can be written

$$G = \{(1)(2)(3)(4), (13)(24), (14)(23), (12)(34)\}$$

A sequence of vertex labels in parentheses like $(abc\ldots r)$ is saying that the motion it represents moves a to b, b to c,$\ldots$, and r to a. Another parenthesis means there is another separate cycle.

A 90° clockwise rotation of a square, with vertices numbered 1 through 4 clockwise, would be represented by (1234).

The stacked notation for permutations is awkward for what we are about to do. The figures-with-arrows is useful, but takes up too much space. So we introduce a compromise called the *disjoint cycle notation*.

In this we take each closed cycle in the points-with-arrows figure and list it (in order of the arrows) in parentheses. Thus our entire group G can be written

$$G = \{(1)(2)(3)(4), (13)(24), (14)(23), (12)(34)\}$$

A sequence of vertex labels in parentheses like $(abc \ldots r)$ is saying that the motion it represents moves a to b, b to c,$\ldots$, and r to a. Another parenthesis means there is another separate cycle.

A 90° clockwise rotation of a square, with vertices numbered 1 through 4 clockwise, would be represented by (1234). The identity always looks like $(1)(2)(3)(4)\cdots$ (actual numbers depending on how many vertices).

The stacked notation for permutations is awkward for what we are about to do. The figures-with-arrows is useful, but takes up too much space. So we introduce a compromise called the *disjoint cycle notation*.

In this we take each closed cycle in the points-with-arrows figure and list it (in order of the arrows) in parentheses. Thus our entire group G can be written

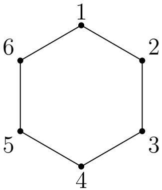
$$G = \{(1)(2)(3)(4), (13)(24), (14)(23), (12)(34)\}$$

A sequence of vertex labels in parentheses like $(abc \dots r)$ is saying that the motion it represents moves a to b, b to c,..., and r to a. Another parenthesis means there is another separate cycle.

A 90° clockwise rotation of a square, with vertices numbered 1 through 4 clockwise, would be represented by (1234). The identity always looks like $(1)(2)(3)(4)\cdots$ (actual numbers depending on how many vertices). On the next page is an example of the 6 rotations of a regular hexagon written in this notation.

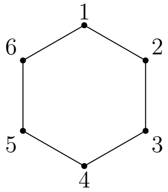
These are (in order): rotation by $0°$, $60°$, $120°$, $180°$, $240°$, and $300°$.

These are (in order): rotation by $0°$, $60°$, $120°$, $180°$, $240°$, and $300°$.



$(1)(2)(3)(4)(5)(6)$, (123456), $(135)(246)$,
$(14)(25)(36)$, $(153)(264)$, (165432)

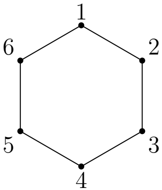
These are (in order): rotation by 0°, 60°, 120°, 180°, 240°, and 300°.



$(1)(2)(3)(4)(5)(6)$, (123456), $(135)(246)$, $(14)(25)(36)$, $(153)(264)$, (165432)

There are 6 reflections of this figure, and they are expressed by $(1)(26)(35)(4)$, $(2)(13)(46)(5)$, $(3)(24)(15)(6)$, $(12)(36)(45)$, $(14)(23)(56)$, $(16)(25)(34)$.

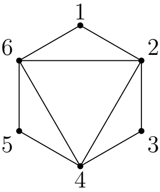
These are (in order): rotation by $0°$, $60°$, $120°$, $180°$, $240°$, and $300°$.

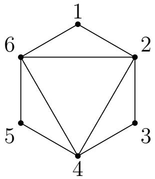


$(1)(2)(3)(4)(5)(6)$, (123456), $(135)(246)$, $(14)(25)(36)$, $(153)(264)$, (165432)

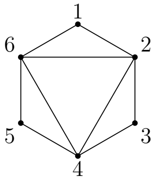
There are 6 reflections of this figure, and they are expressed by $(1)(26)(35)(4)$, $(2)(13)(46)(5)$, $(3)(24)(15)(6)$, $(12)(36)(45)$, $(14)(23)(56)$, $(16)(25)(34)$.

In case it is not clear how to list the rigid motions of a figure using disjoint cycle notation, here is an example completely worked out for the following figure.

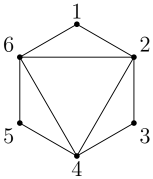




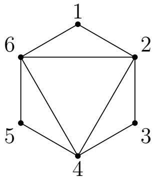
Not all of the rotations of a hexagon are rigid motions of this figure.



Not all of the rotations of a hexagon are rigid motions of this figure. A rotation has to move vertex 1 to vertex 3 or 5 because those are the ones where only two edges meet.

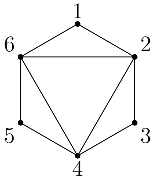


Not all of the rotations of a hexagon are rigid motions of this figure. A rotation has to move vertex 1 to vertex 3 or 5 because those are the ones where only two edges meet. Lines of symmetry (for a hexagon) that connect midpoints of opposite sides are not lines of symmetry for this figure.



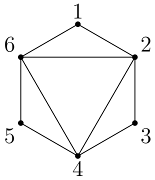
Not all of the rotations of a hexagon are rigid motions of this figure. A rotation has to move vertex 1 to vertex 3 or 5 because those are the ones where only two edges meet. Lines of symmetry (for a hexagon) that connect midpoints of opposite sides are not lines of symmetry for this figure. Rigid motions must keep the triangle unchanged. So, we have

- The identity: $(1)(2)(3)(4)(5)(6)$



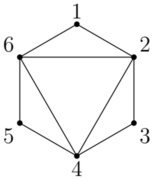
Not all of the rotations of a hexagon are rigid motions of this figure. A rotation has to move vertex 1 to vertex 3 or 5 because those are the ones where only two edges meet. Lines of symmetry (for a hexagon) that connect midpoints of opposite sides are not lines of symmetry for this figure. Rigid motions must keep the triangle unchanged. So, we have

- The identity: $(1)(2)(3)(4)(5)(6)$
- Rotate 120°: $(135)(246)$



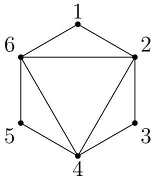
Not all of the rotations of a hexagon are rigid motions of this figure. A rotation has to move vertex 1 to vertex 3 or 5 because those are the ones where only two edges meet. Lines of symmetry (for a hexagon) that connect midpoints of opposite sides are not lines of symmetry for this figure. Rigid motions must keep the triangle unchanged. So, we have

- The identity: $(1)(2)(3)(4)(5)(6)$
- Rotate $120°$: $(135)(246)$
- Rotate $240°$: $(153)(264)$



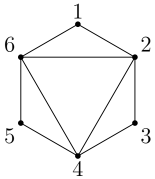
Not all of the rotations of a hexagon are rigid motions of this figure. A rotation has to move vertex 1 to vertex 3 or 5 because those are the ones where only two edges meet. Lines of symmetry (for a hexagon) that connect midpoints of opposite sides are not lines of symmetry for this figure. Rigid motions must keep the triangle unchanged. So, we have

- The identity: $(1)(2)(3)(4)(5)(6)$
- Rotate $120°$: $(135)(246)$
- Rotate $240°$: $(153)(264)$
- Reflect holding 1 and 4 in place: $(1)(26)(35)(4)$



Not all of the rotations of a hexagon are rigid motions of this figure. A rotation has to move vertex 1 to vertex 3 or 5 because those are the ones where only two edges meet. Lines of symmetry (for a hexagon) that connect midpoints of opposite sides are not lines of symmetry for this figure. Rigid motions must keep the triangle unchanged. So, we have

- The identity: $(1)(2)(3)(4)(5)(6)$
- Rotate $120°$: $(135)(246)$
- Rotate $240°$: $(153)(264)$
- Reflect holding 1 and 4 in place: $(1)(26)(35)(4)$
- Reflect holding 2 and 5 in place: $(13)(2)(46)(5)$



Not all of the rotations of a hexagon are rigid motions of this figure. A rotation has to move vertex 1 to vertex 3 or 5 because those are the ones where only two edges meet. Lines of symmetry (for a hexagon) that connect midpoints of opposite sides are not lines of symmetry for this figure. Rigid motions must keep the triangle unchanged. So, we have

- The identity: $(1)(2)(3)(4)(5)(6)$
- Rotate $120°$: $(135)(246)$
- Rotate $240°$: $(153)(264)$
- Reflect holding 1 and 4 in place: $(1)(26)(35)(4)$
- Reflect holding 2 and 5 in place: $(13)(2)(46)(5)$
- Reflect holding 3 and 6 in place: $(15)(24)(3)(6)$

Here are a couple more examples. Take them home, draw the figures and see if you can see how I got them.

Here are a couple more examples. Take them home, draw the figures and see if you can see how I got them.

A regular pentagon with vertices labeled from 1 to 5 clockwise.

Here are a couple more examples. Take them home, draw the figures and see if you can see how I got them.

A regular pentagon with vertices labeled from 1 to 5 clockwise. The identity: $(1)(2)(3)(4)(5)$.

Here are a couple more examples. Take them home, draw the figures and see if you can see how I got them.

A regular pentagon with vertices labeled from 1 to 5 clockwise. The identity: $(1)(2)(3)(4)(5)$. Rotate one position clockwise: (12345).

Here are a couple more examples. Take them home, draw the figures and see if you can see how I got them.

A regular pentagon with vertices labeled from 1 to 5 clockwise. The identity: (1)(2)(3)(4)(5). Rotate one position clockwise: (12345). Rotate two positions: (13524).

Here are a couple more examples. Take them home, draw the figures and see if you can see how I got them.

A regular pentagon with vertices labeled from 1 to 5 clockwise. The identity: (1)(2)(3)(4)(5). Rotate one position clockwise: (12345). Rotate two positions: (13524). Rotate three positions: (14253).

Here are a couple more examples. Take them home, draw the figures and see if you can see how I got them.

A regular pentagon with vertices labeled from 1 to 5 clockwise. The identity: (1)(2)(3)(4)(5). Rotate one position clockwise: (12345). Rotate two positions: (13524). Rotate three positions: (14253). Rotate four positions: (15432).

Here are a couple more examples. Take them home, draw the figures and see if you can see how I got them.

A regular pentagon with vertices labeled from 1 to 5 clockwise. The identity: (1)(2)(3)(4)(5). Rotate one position clockwise: (12345). Rotate two positions: (13524). Rotate three positions: (14253). Rotate four positions: (15432). And the following 5 reflections, where each is determined by a line of symmetry from a vertex to the middle of the opposite side: (1)(25)(34), (2)(13)(45), (3)(24)(15), (4)(35)(12), and (5)(14)(23).

Here are a couple more examples. Take them home, draw the figures and see if you can see how I got them.

A regular pentagon with vertices labeled from 1 to 5 clockwise. The identity: (1)(2)(3)(4)(5). Rotate one position clockwise: (12345). Rotate two positions: (13524). Rotate three positions: (14253). Rotate four positions: (15432). And the following 5 reflections, where each is determined by a line of symmetry from a vertex to the middle of the opposite side: (1)(25)(34), (2)(13)(45), (3)(24)(15), (4)(35)(12), and (5)(14)(23).

I should point out that there is some choice in how each cycle is written.

Here are a couple more examples. Take them home, draw the figures and see if you can see how I got them.

A regular pentagon with vertices labeled from 1 to 5 clockwise. The identity: (1)(2)(3)(4)(5). Rotate one position clockwise: (12345). Rotate two positions: (13524). Rotate three positions: (14253). Rotate four positions: (15432). And the following 5 reflections, where each is determined by a line of symmetry from a vertex to the middle of the opposite side: (1)(25)(34), (2)(13)(45), (3)(24)(15), (4)(35)(12), and (5)(14)(23).

I should point out that there is some choice in how each cycle is written. That is, (12345) is the same permutation as (34512) because both mean $1 \to 2 \to 3 \to 4 \to 5 \to 1$.

Here are a couple more examples. Take them home, draw the figures and see if you can see how I got them.

A regular pentagon with vertices labeled from 1 to 5 clockwise. The identity: (1)(2)(3)(4)(5). Rotate one position clockwise: (12345). Rotate two positions: (13524). Rotate three positions: (14253). Rotate four positions: (15432). And the following 5 reflections, where each is determined by a line of symmetry from a vertex to the middle of the opposite side: (1)(25)(34), (2)(13)(45), (3)(24)(15), (4)(35)(12), and (5)(14)(23).

I should point out that there is some choice in how each cycle is written. That is, (12345) is the same permutation as (34512) because both mean $1 \to 2 \to 3 \to 4 \to 5 \to 1$.

A square attached at the center to an axle that allows it to rotate but prevents from being flipped over:

Here are a couple more examples. Take them home, draw the figures and see if you can see how I got them.

A regular pentagon with vertices labeled from 1 to 5 clockwise. The identity: (1)(2)(3)(4)(5). Rotate one position clockwise: (12345). Rotate two positions: (13524). Rotate three positions: (14253). Rotate four positions: (15432). And the following 5 reflections, where each is determined by a line of symmetry from a vertex to the middle of the opposite side: (1)(25)(34), (2)(13)(45), (3)(24)(15), (4)(35)(12), and (5)(14)(23).

I should point out that there is some choice in how each cycle is written. That is, (12345) is the same permutation as (34512) because both mean $1 \to 2 \to 3 \to 4 \to 5 \to 1$.

A square attached at the center to an axle that allows it to rotate but prevents from being flipped over: This group has 4 rotations. If the vertices are labeled from 1 to 4 clockwise, they have the following disjoint cycle representations: (1)(2)(3)(4), (1234), (13)(24), (1432).

The cycle structure of the permutations of the vertices will be used to compute the number of distinguishable colorings.

The cycle structure of the permutations of the vertices will be used to compute the number of distinguishable colorings. To get to where we can do that, we can change our point of view about rigid motions.

The cycle structure of the permutations of the vertices will be used to compute the number of distinguishable colorings. To get to where we can do that, we can change our point of view about rigid motions. They move the figure and permute the vertices, but if we think of what they do to the colored figures, they also permute those.

The cycle structure of the permutations of the vertices will be used to compute the number of distinguishable colorings. To get to where we can do that, we can change our point of view about rigid motions. They move the figure and permute the vertices, but if we think of what they do to the colored figures, they also permute those.

So let's remind ourselves what we are trying to count.

The cycle structure of the permutations of the vertices will be used to compute the number of distinguishable colorings. To get to where we can do that, we can change our point of view about rigid motions. They move the figure and permute the vertices, but if we think of what they do to the colored figures, they also permute those.

So let's remind ourselves what we are trying to count. If we keep a figure unmoved and paint its vertices one of two colors we have 2^m possible colorings (or "configurations"), where m is the number of vertices.

The cycle structure of the permutations of the vertices will be used to compute the number of distinguishable colorings. To get to where we can do that, we can change our point of view about rigid motions. They move the figure and permute the vertices, but if we think of what they do to the colored figures, they also permute those.

So let's remind ourselves what we are trying to count. If we keep a figure unmoved and paint its vertices one of two colors we have 2^m possible colorings (or "configurations"), where m is the number of vertices. For our triangle/hexagon example there are 2^6 possibilities. If there are 3 colors to choose from, then there are 3^6 possibilities.

We consider two of these colorings *equivalent* if some rigid motion of the figure turns one into the other.

The cycle structure of the permutations of the vertices will be used to compute the number of distinguishable colorings. To get to where we can do that, we can change our point of view about rigid motions. They move the figure and permute the vertices, but if we think of what they do to the colored figures, they also permute those.

So let's remind ourselves what we are trying to count. If we keep a figure unmoved and paint its vertices one of two colors we have 2^m possible colorings (or "configurations"), where m is the number of vertices. For our triangle/hexagon example there are 2^6 possibilities. If there are 3 colors to choose from, then there are 3^6 possibilities.

We consider two of these colorings *equivalent* if some rigid motion of the figure turns one into the other. This collects the colorings (as in the book on page 780) into *equivalence classes* where the colorings in each class are equivalent to each other,

The cycle structure of the permutations of the vertices will be used to compute the number of distinguishable colorings. To get to where we can do that, we can change our point of view about rigid motions. They move the figure and permute the vertices, but if we think of what they do to the colored figures, they also permute those.

So let's remind ourselves what we are trying to count. If we keep a figure unmoved and paint its vertices one of two colors we have 2^m possible colorings (or "configurations"), where m is the number of vertices. For our triangle/hexagon example there are 2^6 possibilities. If there are 3 colors to choose from, then there are 3^6 possibilities.

We consider two of these colorings *equivalent* if some rigid motion of the figure turns one into the other. This collects the colorings (as in the book on page 780) into *equivalence classes* where the colorings in each class are equivalent to each other, but not equivalent to any coloring in a different class.

The cycle structure of the permutations of the vertices will be used to compute the number of distinguishable colorings. To get to where we can do that, we can change our point of view about rigid motions. They move the figure and permute the vertices, but if we think of what they do to the colored figures, they also permute those.

So let's remind ourselves what we are trying to count. If we keep a figure unmoved and paint its vertices one of two colors we have 2^m possible colorings (or "configurations"), where m is the number of vertices. For our triangle/hexagon example there are 2^6 possibilities. If there are 3 colors to choose from, then there are 3^6 possibilities.

We consider two of these colorings *equivalent* if some rigid motion of the figure turns one into the other. This collects the colorings (as in the book on page 780) into *equivalence classes* where the colorings in each class are equivalent to each other, but not equivalent to any coloring in a different class.

We seek to find the number of equivalence classes.

What allows us to do this is Burnside's Theorem.

What allows us to do this is Burnside's Theorem. Its setting is a group G (in our case the rigid motions) whose elements produce permutations of a set $\mathscr{C}$ of configurations.

What allows us to do this is Burnside's Theorem. Its setting is a group G (in our case the rigid motions) whose elements produce permutations of a set $\mathscr{C}$ of configurations. If g is an element of G, we let g^* be the 1-1 function on $\mathscr{C}$ that g produces.

What allows us to do this is Burnside's Theorem. Its setting is a group G (in our case the rigid motions) whose elements produce permutations of a set $\mathscr{C}$ of configurations. If g is an element of G, we let g^* be the 1-1 function on $\mathscr{C}$ that g produces. Looking back at the first figure (squares with two red and two white vertices) and consider that motion g that rotates the square 90°.

What allows us to do this is Burnside's Theorem. Its setting is a group G (in our case the rigid motions) whose elements produce permutations of a set $\mathscr{C}$ of configurations. If g is an element of G, we let g^* be the 1-1 function on $\mathscr{C}$ that g produces. Looking back at the first figure (squares with two red and two white vertices) and consider that motion g that rotates the square 90°. Then g^* turns every coloring into another, and in particular g^* turns the first colored square in that figure into the second.

What allows us to do this is Burnside's Theorem. Its setting is a group G (in our case the rigid motions) whose elements produce permutations of a set $\mathscr{C}$ of configurations. If g is an element of G, we let g^* be the 1-1 function on $\mathscr{C}$ that g produces. Looking back at the first figure (squares with two red and two white vertices) and consider that motion g that rotates the square 90°. Then g^* turns every coloring into another, and in particular g^* turns the first colored square in that figure into the second.

Let $\psi(g^*)$ be the number of configurations in $\mathscr{C}$ that g^* leaves unchanged.

What allows us to do this is Burnside's Theorem. Its setting is a group G (in our case the rigid motions) whose elements produce permutations of a set $\mathscr{C}$ of configurations. If g is an element of G, we let g^* be the 1-1 function on $\mathscr{C}$ that g produces. Looking back at the first figure (squares with two red and two white vertices) and consider that motion g that rotates the square 90°. Then g^* turns every coloring into another, and in particular g^* turns the first colored square in that figure into the second.

Let $\psi(g^*)$ be the number of configurations in $\mathscr{C}$ that g^* leaves unchanged. For example, if g is the 90° rotation of the square the only colorings unchanged are those where every vertex is the same color.

What allows us to do this is Burnside's Theorem. Its setting is a group G (in our case the rigid motions) whose elements produce permutations of a set $\mathscr{C}$ of configurations. If g is an element of G, we let g^* be the 1-1 function on $\mathscr{C}$ that g produces. Looking back at the first figure (squares with two red and two white vertices) and consider that motion g that rotates the square 90°. Then g^* turns every coloring into another, and in particular g^* turns the first colored square in that figure into the second.

Let $\psi(g^*)$ be the number of configurations in $\mathscr{C}$ that g^* leaves unchanged. For example, if g is the 90° rotation of the square the only colorings unchanged are those where every vertex is the same color. For the identity, all colorings are unchanged.

Theorem (Burnside's Theorem)

The number of equivalence classes of color configurations is

$$\frac{1}{|G|}\sum_{g\in G}\psi(g^*)$$