The Rings $\mathbb{Z}_n$

Daniel H. Luecking

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If R is a ring and n is any positive integer we can create a new ring called $M_n(R)$ whose elements are all the $n \times n$ matrices whose entries are elements of R. The zero of $M_n(R)$ is the matrix with all zero entries. Even if R is commutative, $M_n(R)$ almost never is if $n > 1$. If R has a unity, then so does $M_n(R)$. $M_n(R)$ always has proper zero divisors when $n > 1$ unless $R = \{0\}$.

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Theorem

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Definition

If k and n are two positive integers then a positive integer d is called a *common divisor* of k and n iff d evenly divides both k and n. The largest common divisor is denoted $\gcd(k, n)$.

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In fact, factoring large numbers is one of the hardest problems in computing (by 'large', I mean having thousands of bits in base 2).

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Finally, $\gcd(6, 3) = 3$ because 3 divides 6 evenly.

This tells us that
$3 = \gcd(6,3) = \gcd(15,6) = \gcd(36,15) = \gcd(195,36)$.

Here is the whole process condensed:

$$\begin{aligned} 195 &= 5 \cdot 36 + 15 \\ 36 &= 2 \cdot 15 + 6 \\ 15 &= 2 \cdot 6 \quad + 3 \\ 6 &= 2 \cdot 3 \quad + 0 \end{aligned}$$

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Theorem

An element k of $\mathbb{Z}_n$ is a unit if and only if $\gcd(n, k) = 1$. It is a proper zero divisor if and only if it is not zero and $\gcd(n, k) > 1$.

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Let's illustrate the other half of the theorem. Consider finding the inverse of 7 in $\mathbb{Z}_{73}$. Lets first check that $\gcd(73, 7) = 1$:

$$\begin{aligned} 73 &= 10 \cdot 7 + 3 \\ 7 &= \ \ 2 \cdot 3 + 1 \\ 3 &= \ \ 3 \cdot 1 + 0 \end{aligned}$$

That is $\gcd(73, 7) = 1$.

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This leads to

$$k = 2n - 20k + r_2 \;\text{ or }\; 21k - 2n = r_2$$

Since $r_2 = 1$, $k = 7$ and $n = 73$, this becomes $21(7) = 2(73) + 1$. This tells us that $21 \cdot 7 = 21(7) \bmod 73 = 1$. By definition, $7^{-1} = 21$ in $\mathbb{Z}_{73}$.

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Here's another example: Find the inverse of 34 in the ring $\mathbb{Z}_{371}$ (or else prove it has no inverse).

Here's the Euclidean algorithm:

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\begin{aligned}
371 &= 10 \cdot 34 + 31 \\
34 &= 1 \cdot 31 + 3 \\
31 &= 10 \cdot 3 + 1
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We can skip the division by 1 because the remainder will always be 0.

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This time we need to eliminate r_1 and r_2 and leave r_3 as a combination of n and k. We can do this like before: put $r_1 = n - 10k$ into the second and third equations. Then use the second equation to get a formula for r_2 and put that in the third equation.

If you've had Linear Algebra you can rewrite this as

$$\begin{aligned} r_1 \qquad\qquad\quad &= n - 10k \\ r_1 + \quad r_2 \qquad &= k \\ -r_1 + 10r_2 + r_3 &= 0 \end{aligned}$$

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$$\begin{aligned} r_1 \qquad\qquad\qquad &= n - 10k \\ + \quad r_2 \qquad\quad &= -n + 11k \\ + 10r_2 + r_3 &= n - 10k \end{aligned}$$

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Now subtract 10 times equation 2 from equation 3 to get

$$\begin{aligned} r_1 &= n - 10k \\ r_2 &= -n + 11k \\ r_3 &= 11n - 120k \end{aligned}$$

The last equation says that $1 = (-120)(34) + 11(371)$. This tells us that $(-120) \cdot 34 = 1$ in $\mathbb{Z}_{371}$. Thus $34^{-1} = -120 = 251$

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For Linear Algebra aficionados only: Use the augmented matrix

$$\left(\begin{array}{ccc|cc} r_1 & r_2 & r_3 & n & k \\ \hline 1 & 0 & 0 & 1 & -10 \\ 1 & 1 & 0 & 0 & 1 \\ -1 & 10 & 1 & 0 & 0 \end{array}\right)$$

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Then read off $1 = 11n + (-120)k$.