Rings and Things

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In higher mathematics we study systems that consist of a set on which one or more binary operations are defined. The above description gives us a set, namely $\{0, 1, 2, 3, \ldots, 23\}$ and an operation '$+$'. The operation is analogous to addition, but is not the usual operation of addition of integers. Let us call it $\hat{+}$ (temporarily). Its formal definition is

For any x and y in $\{0, 1, 2, \ldots, 23\}$, let $x \mathbin{\hat{+}} y = (x + y) \bmod 24$.

To make sense of this we need to know what mod means.

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For example, Since $28 = 1 \cdot 24 + 4$, then for $k = 28$ and $n = 24$ we have $28 \bmod 24 = 4$. Similarly. $71 = 2 \cdot 24 + 23$ so $71 \bmod 24 = 23$ and $72 \bmod 24 = 0$.

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By definition, we always have $0 \leq k \bmod n \leq n - 1$. We can obtain $k \bmod m$ by second grade divison: To find, for example $68 \bmod 9 = 5$ we say "9 goes into 68 seven times (for 63) with a remainder of 5." Here's an example computing $721 \bmod 101 = 14$:

$$\begin{array}{r|r}
 & 7 \text{ R } 14 \\
\hline
101 & 721 \\
 & 707 \\
\cline{2-2}
 & 14
\end{array}$$

We can do modular arithmetic in any 'base': Define $\mathbb{Z}_n = \{0, 1, 2, \cdots, n-1\}$ and define 'addition' on $\mathbb{Z}_n$ by $x \hat{+} y = (x + y) \bmod n$.

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So in $\mathbb{Z}_6 = \{0, 1, 2, 3, 4, 5\}$ we have $5 \mathbin{\hat{+}} 4 = 3$ (because $9 \bmod 6 = 3$) and $4 \mathbin{\hat{\cdot}} 5 = 2$ (because $20 \bmod 6 = 2$).

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This means that $a - b$ is evenly divisible by n. The notation we will are using: $a = (b \bmod n)$ or $a = (b \,\%\, n)$ means two things

$$a \equiv b \pmod{n} \quad \text{and} \quad 0 \leq a < n\,.$$

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The sets $\mathbb{Z}_n$ with the operations of 'addition modulo n' and 'multiplication modulo n' are all examples of rings. Notice that in $\mathbb{Z}_{24}$ we saw that $17 \mathbin{\hat{+}} 7 = 0$. By part A4, that means that $-17=7$ and $-7=17$.

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- A5 can be extended to any number of elements: how they are grouped does not change the result of the multiplication.
- A6 applies to any length of the sum:

$$x \cdot (y_1 + y_2 + \cdots + y_n) = x \cdot y_1 + x \cdot y_2 + \cdots + x \cdot y_n$$

  and the same for multiplying on the right.

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The ring of integers has a special element, the number 1, that satisfies $1 \cdot x = x$ for every element x. This not always true: the set of even integers with the usual operations of addition and multiplication is a ring, but has no such element.

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- If there exists an element $u \neq 0$ of R that satisfies $u \cdot x = x = x \cdot u$ for every x in R, then u is called a *unity* or a *multiplicative identity* (or just 'the identity') and we say R is a *ring with unity*.

Definition

An element $x \neq 0$ of a ring R is called a *proper zero divisor* if there is another element $y \neq 0$ such that $x \cdot y = 0$.

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Definition

Let $(R, +, \cdot)$ be a ring with unity u. If x is in R and there is an element y in R such that $x \cdot y = u = y \cdot x$ we call y the *multiplicative inverse* of x. In that case we say that x is a *unit* (or *is invertible*) and we call its multiplicative inverse x^{-1}.

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In $\mathbb{Z}_6$ we have 1 for the unity. The elements 1 and 5 are units: since $1 \cdot 1 = 1$ and $5 \cdot 5 = 1$ it follows that each is its own multiplicative inverse.

In $\mathbb{Z}_{15}$ we have units 2 and 8 (inverses of each other), 7 and 13 (inverses of each other) and also 1, 4, 11 and 14 (each is its own inverse).

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The next simplest might be $\mathbb{Z}_2$. In applications, 0 often represents 'false' and 1 represents 'true'. Then multiplication represents the AND operation and addition represents XOR (the 'exclusive or' operation). This is a commutative ring with unity with no proper zero divisors.

$+$	0	1
0	0	1
1	1	0

$\cdot$	0	1
0	0	0
1	0	1

Addition and multiplication tables for $\mathbb{Z}_2$

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Proof: If u is the unity then $u \cdot u = u$.

If we multiply $(-x) \cdot (-x^{-1})$ we get $x \cdot x^{-1} = u$ (law of signs). The other order is similar.

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We can do repeated multiplications as well: the inverse of $8 \cdot 13 \cdot 13 = 2$ is $7 \cdot 7 \cdot 2 = 8$.