Recurrence Relations

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Thus $\alpha + \beta i = \rho(\cos \theta + i \sin \theta)$. A famous theorem due to Euler says that

$$(\alpha + \beta i)^n = \rho^n (\cos \theta + i \sin \theta)^n = \rho^n (\cos(n\theta) + i \sin(n\theta))$$

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$$a_n = (2\sqrt{2})^n[C_1 \cos(45n) + C_2 \sin(45n)]$$

The initial conditions of that example ($a_0 = 2$, $a_1 = 12$) give the following equation for C_1 and C_2 (note that $\cos 0 = 1$, $\sin 0 = 0$, $\cos 45^\circ = \sqrt{2}/2$ and $\sin 45^\circ = \sqrt{2}/2$)

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Another example with complex roots, which I will process both ways:

$$\begin{gathered} a_n - 2a_{n-1} + 5a_{n-2} = 0, \quad n \geq 0 \\ a_0 = 0,\ a_1 = 3 \end{gathered}$$

Characteristic equation $r^2 - 2r + 5 = 0$, with roots $1 \pm 2i$.

Complex powers method: General solution
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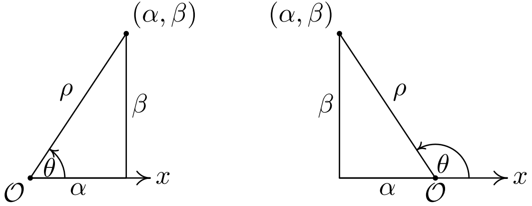
Something to note about the trigonometric function method: These two pictures represent the ρ and θ for two examples. The first is where α is positive and so the point (α, β) is to the right of the y axis The second is where α is negative and so the point (α, β) is to the left of the y axis



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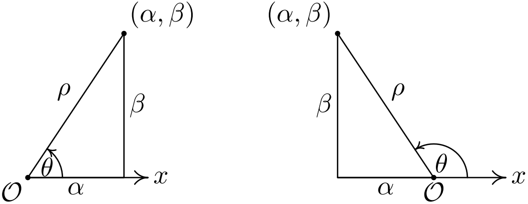

Because $\rho\cos\theta = \alpha$ and $\rho\sin\theta = \beta$ always hold, the initial conditions for $a_n = \rho^n[C_1\cos(n\theta) + C_2\sin(n\theta)]$ always simplify to

$$\begin{aligned} C_1 &= a_0 \\ \alpha C_1 + \beta C_2 &= a_1 \end{aligned}$$

Nonhomogeneous recurrence relations

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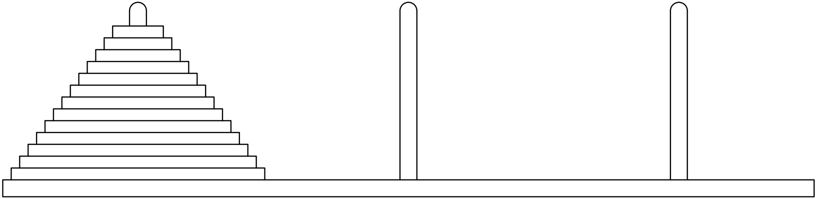
Here is a picture of a possible starting point with 13 disks:



The goal is to move all the disks from the first pole to the last.

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- It is not allowed to place a disk on top of a smaller disk.
- The game ends when all the disks are on the third pole

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That is

$$a_n = 2a_{n-1} + 1, \quad n \geq 1$$
$$a_0 = 0.$$

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This has characteristic equation $r - 2 = 0$ with the root $r = 2$ and so the general solution is $a_n = C_1 2^n$. However, this doesn't solve the original nonhomogeneous equation.

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This exemplifies a general rule: *to find the general solution of a nonhomogeneous recurrence relation, just find one solution and add it to the general solution of the associated homogeneous recurrence relation.*

To see how we can use this, we examine our recurrence relation:

$$a_n - 2a_{n-1} = 1$$

and remark that since the right side is constant, we need the left side to be constant, so perhaps letting a_n be a constant will give us a solution.

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Following our previous discussion, we know that all other solutions have the form $a_n = C_1 2^n - 1$. Now that we have the general solution, we can impose the initial condition to find C_1:

$$C_1 2^0 - 1 = 0 \quad \text{or} \quad C_1 = 1.$$

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$$a_n - 4a_{n-1} + 3a_{n-2} = (3)2^n, \quad n \geq 2$$

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Step 1: Find the general solution of the homogeneous version:

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A significant step in the solution was to seemingly guess what the form of a solution might be. That is, we guessed that the solution should look something like the right hand side of the recurrence relation.

This actually works surprisingly often. As an example, lets work through the following. For the moment, lets not even mention initial conditions.

$$a_n - 4a_{n-1} + 3a_{n-2} = (3)2^n, \quad n \geq 2$$

Step 1: Find the general solution of the homogeneous version:

$$a_n - 4a_{n-1} + 3a_{n-2} = 0, \quad n \geq 2$$

To keep straight which expressions solve which equations, we'll call this the *homogeneous solution* and denote it by $a_n^{(h)}$.

The characteristic equation $r^2 - 4r + 3 = 0$ has roots 3 and 1 so

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We examine the right side of our recurrence relation, $(3)2^n$, and reason as follows: if we substitute a constant times 2^n for a_n then the left side will produce three terms with 2^n in each, so this will have a chance of adding up to $(3)2^n$.

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Thus, we should set $a_n = A2^n$ and run this through the equation to see what A should be.

If $a_n = A2^n$ then $a_{n-1} = A2^{n-1}$ and $a_{n-2} = A2^{n-2}$. Putting these in $a_n - 4a_{n-1} + 3a_{n-2} = (3)2^n$ gives

$$
\begin{aligned}
A2^n - 4A2^{n-1} + 3A2^{n-2} &= (3)2^n \\
[A - 4A2^{-1} + 3A2^{-2}]2^n &= (3)2^n \\
A - 2A + (3/4)A &= 3 \\
-(1/4)A &= 3 \\
A &= -12
\end{aligned}
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Step 3: Add the 2 parts together to get the general solution. The general solution is $a_n = a_n^{(h)} + a_n^{(p)} = C_1 3^n + C_2 - (12)2^n$.

When initial conditions are present there is a final step.

Step 4: Use the initial conditions to find, and then fill in, the constants.

Lets illustrate this for the current problem. Here are some simple initial conditions

$$a_n - 4a_{n-1} + 3a_{n-2} = (3)2^n, \quad n \geq 2$$
$$a_0 = 0, \ a_1 = 0.$$

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We obtain C_1 and C_2 by putting $n = 0$ and 1 into our general solution $a_n = C_1 3^n + C_2 - (12)2^n$:

$$C_1 + C_2 - 12 = 0$$
$$3C_1 + C_2 - 24 = 0$$

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Therefore, the completed solution is $a_n = (6)3^n + 6 - (12)2^n$.