# Recurrence Relations

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$$C_1 + C_2 = 1$$
$$\left( \frac{1 + \sqrt{5}}{2} \right) C_1 + \left( \frac{1 - \sqrt{5}}{2} \right) C_2 = 2$$

Here, the hard part is solving for the $C$'s The first equation, $C_1 + C_2 = 1$ is not complicated. The second equation can be rewritten as

$$(C_1 + C_2)\left(\frac{1}{2}\right) + (C_1 - C_2)\left(\frac{\sqrt{5}}{2}\right) = 2$$

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This we can multiply by $2/\sqrt{5}$ to get

$$C_1 - C_2 = \frac{3}{\sqrt{5}} = \frac{3\sqrt{5}}{5}$$

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and $C_2$ we get by subtracting and dividing by 2:

$$C_2 = \frac{1}{2} - \frac{3\sqrt{5}}{10}$$

A third order example:

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Finding the $C$'s:

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Solving gives $C_1 = 0$, $C_2 = -1/5$, $C_3 = 1/5$ so $a_n = (-1/5)(-2)^n + (1/5)3^n$

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But $C_1$ cannot be both 1 and $4/3$.

If we tried $a_n = C_1 3^n + C_2 3^n$ then we would get

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$$a_n = C_1 3^n + C_2\, n\, 3^n$$

[For order 3 recurrence relation it is possible to have triple roots (and still higher repetitions for higher orders). In that case, multiply by $n$ again to get basic solutions $r^n$, $nr^n$ and $n^2 r^n$ (if $r$ is a triple root).]

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Here is another example:

$$\begin{aligned} a_n - 2a_{n-1} + a_{n-2} &= 0, \quad n \geq 2 \\ a_0 = 2,\ a_1 &= 5 \end{aligned}$$

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From $r^2 - 6r + 10 = 0$ the quadratic formula gives

$$\begin{aligned} r &= \frac{6 \pm \sqrt{6^2 - 4(10)}}{2} = \frac{6 \pm \sqrt{-4}}{2} \\ &= \frac{6 \pm \sqrt{4(-1)}}{2} = \frac{6 \pm 2\sqrt{-1}}{2} \\ &= 3 \pm i \end{aligned}$$

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So we can just use these roots as we would any real roots and get the general solution

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Since $3(C_1 + C_2) = 3$, this gives us

$$(C_1 - C_2)i = 2 \quad \text{or} \quad C_1 - C_2 = 2/i$$

So the system of equations becomes

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Since $(-i)(i) = 1$ we have $1/i = -i$ and so the above solution can be rewritten in the more standard form

$$a_n = (1/2 - i)(3 + i)^n + (1/2 + i)(3 - i)^n$$