Recurrence Relations

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Combinatorics problems that lead to recurrence relations

The chip stacking problem

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An equivalent problem is: how many different bitstrings (strings consisting only of 0's and 1's) of length n contain no consecutive 1's?

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So the sequence a_n looks like $1, 2, 3, 5, 8, 13, \ldots$.

Our next section will give us the tools to find the formula

$$a_n = \frac{5 + 3\sqrt{5}}{10} \left(\frac{1 + \sqrt{5}}{2} \right)^n + \frac{5 - 3\sqrt{5}}{10} \left(\frac{1 - \sqrt{5}}{2} \right)^n$$

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$$\begin{aligned} a_n &= 3a_{n-1} \\ a_n &= 2a_{n-1} - 5n^2 a_{n-2} \\ a_n &= 4a_{n-1} + 2a_{n-2} + 5a_{n-3} \end{aligned}$$

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where b, c and h are explicit functions of n that make no reference to any terms of the unknown sequence a_k. If the function $h(n)$ on the right side is just 0, the recurrence relation is called *homogeneous*.

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Theorem

If $a_n = f(n)$ is a solutions of a homogeneous linear recurrence relation, then for any constant C so is $a_n = Cf(n)$. If $a_n = g(n)$ is also a solution, then so is $a_n = f(n) + g(n)$.

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For example, consider

$$a_n - 5a_{n-1} + 6a_{n-2} = 0$$

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which is homogeneous and linear. Let us check that both $a_n = 3^n$ and $a_n = 2^n$ are solutions.

For the first, we simply put

$$a_n = 3^n, \quad a_{n-1} = 3^{n-1}, \quad a_{n-2} = 3^{n-2}$$

into the recurrence relation to get

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We can rearrange the left side:

$$(3^2 - 5 \cdot 3^1 + 6)3^{n-2} = 0 \quad \text{or} \quad (9 - 5 \cdot 3 + 6)3^{n-2} = 0$$

and clearly the last one says $0 \cdot 3^{n-2} = 0$, which is true for every $n \geq 2$.

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With these solutions we can build new solutions: $a_n = 2(3^n)$ and $a_n = -(2^n)$ and $a_n = 2(3^n) - (2^n)$. Note that if we add initial conditions $a_0 = 1$ and $a_1 = 4$, then only the last of these satisfies them both.

A further part of the theory of recurrence relations is the following

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For any homogeneous linear recurrence relation of order k *there exist a basic set of solutions* $a_n = f_1(n)$, $a_n = f_2(n)$,$\ldots$,$a_n = f_k(n)$

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For our example $a_n - 5a_{n-1} + 6a_{n-2}$ the basic solutions are $a_n = 3^n$ and $a_n = 2^n$. Thus, all solutions look like $a_n = C_1 3^n + C_2 2^n$. This is called a *general solution*: It satisfies the recurrence relation for any choice of C_1 and C_2, but satisfies any given initial condition for only one choice.

If the general solution is $a_n = C_1 3^n + C_2 2^n$ and the initial conditions are $a_0 = 1$ and $a_1 = 4$ then putting $n = 0$ and $n = 1$ into the solution gives us

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This gives us the general scheme for solving homogeneous linear recurrence relations:

- Find the basic set of solutions.
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- Solve for the constants using the initial conditions.

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For higher order equations one might speculate that one or more of the basic solutions has a similar structure. That is, we might suppose that one solution is a geometric series $a_n = r^n$ for some r.

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If $a_n = r^n$ is to be a solution then r must be a root of this equation:

$$r = \frac{-b \pm \sqrt{b^2 - 4c}}{2} \tag{1}$$

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[Note that $(-2)^n$ is not -2^n. The sequence $(-2)^n$ is $1, -2, 4, -8, 16, \ldots$ while -2^n is $-1, -2, -4, -8, -16, \ldots$.]

Continuing with the same example, let's give it initial conditions:

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From the general solution $a_n = C_1 3^n + C_2(-2)^n$ and the initial conditions we get equations for C_1 and C_2:

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which gives $C_1 = 1$ and $C_2 = -1$ so that $a_n = 3^n - (-2)^n$ is the solution to the initial value problem.

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Here we get $r^2 - 4 = 0$ and roots 2 and -2. The general solution is $a_n = C_1 2^n + C_2(-2)^n$. From

$$\begin{aligned} C_1 \ &+ C_2 = 1 \\ 2C_1 &- 2C_2 = 1 \end{aligned}$$

we get $C_1 = 3/4$ and $C_2 = 1/4$ so that $a_n = (3/4)2^n + (1/4)(-2)^n$.