Forbidden Positions

Daniel H. Luecking

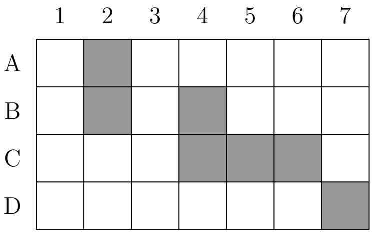
September 11, 2023

Let us attack N first.

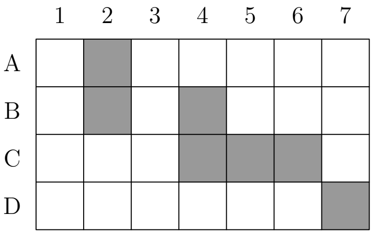
Let us attack N first. This represents the number of ways to place check marks without imposing conditions, that is, it is the number of ways to seat 4 people in 7 seats with no restrictions.

Let us attack N first. This represents the number of ways to place check marks without imposing conditions, that is, it is the number of ways to seat 4 people in 7 seats with no restrictions. We computed this earlier:
$N = P(7, 4) = 7!/3!$.

Let us attack N first. This represents the number of ways to place check marks without imposing conditions, that is, it is the number of ways to seat 4 people in 7 seats with no restrictions. We computed this earlier: $N = P(7, 4) = 7!/3!$.

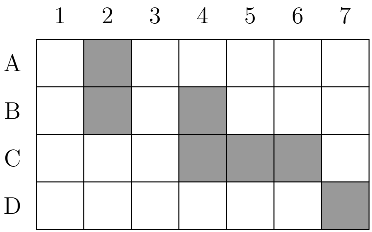


Let us attack N first. This represents the number of ways to place check marks without imposing conditions, that is, it is the number of ways to seat 4 people in 7 seats with no restrictions. We computed this earlier: $N = P(7,4) = 7!/3!$.



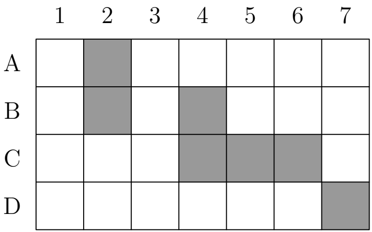
Now let's compute $N(c_1)$. A seating assignment that puts A in a forbidden seat can be obtained in 2 steps: Put A in the forbidden seat (just 1 way) and then assign the seats to the remaining 3 people.

Let us attack N first. This represents the number of ways to place check marks without imposing conditions, that is, it is the number of ways to seat 4 people in 7 seats with no restrictions. We computed this earlier: $N = P(7,4) = 7!/3!$.



Now let's compute $N(c_1)$. A seating assignment that puts A in a forbidden seat can be obtained in 2 steps: Put A in the forbidden seat (just 1 way) and then assign the seats to the remaining 3 people. Since there are 6 seats remaining, this can be done in $P(6,3)$ ways. By the rule of product, $N(c_1) = 1 \cdot P(6,3)$.

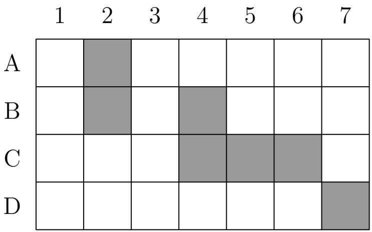
Let us attack N first. This represents the number of ways to place check marks without imposing conditions, that is, it is the number of ways to seat 4 people in 7 seats with no restrictions. We computed this earlier: $N = P(7, 4) = 7!/3!$.



Now let's compute $N(c_1)$. A seating assignment that puts A in a forbidden seat can be obtained in 2 steps: Put A in the forbidden seat (just 1 way) and then assign the seats to the remaining 3 people. Since there are 6 seats remaining, this can be done in $P(6, 3)$ ways. By the rule of product, $N(c_1) = 1 \cdot P(6, 3)$.

For $N(c_2)$ we get 2 ways to do the first step (2 forbidden seats) and again $P(6, 3)$ ways for the second, so $N(c_2) = 2 \cdot P(6, 3)$.

Let us attack N first. This represents the number of ways to place check marks without imposing conditions, that is, it is the number of ways to seat 4 people in 7 seats with no restrictions. We computed this earlier: $N = P(7,4) = 7!/3!$.

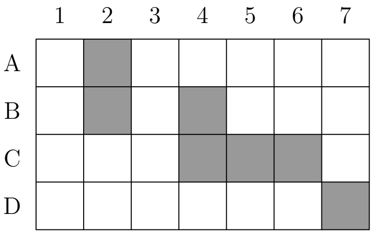


Now let's compute $N(c_1)$. A seating assignment that puts A in a forbidden seat can be obtained in 2 steps: Put A in the forbidden seat (just 1 way) and then assign the seats to the remaining 3 people. Since there are 6 seats remaining, this can be done in $P(6,3)$ ways. By the rule of product, $N(c_1) = 1 \cdot P(6,3)$.

For $N(c_2)$ we get 2 ways to do the first step (2 forbidden seats) and again $P(6,3)$ ways for the second, so $N(c_2) = 2 \cdot P(6,3)$.

Similarly, $N(c_3) = 3 \cdot P(6,3)$ and $N(c_4) = 1 \cdot P(6,3)$.

Let us attack N first. This represents the number of ways to place check marks without imposing conditions, that is, it is the number of ways to seat 4 people in 7 seats with no restrictions. We computed this earlier: $N = P(7,4) = 7!/3!$.



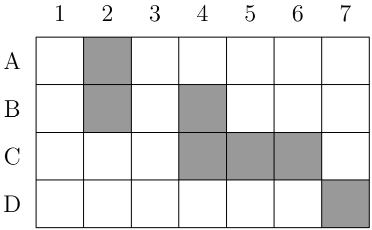
Now let's compute $N(c_1)$. A seating assignment that puts A in a forbidden seat can be obtained in 2 steps: Put A in the forbidden seat (just 1 way) and then assign the seats to the remaining 3 people. Since there are 6 seats remaining, this can be done in $P(6,3)$ ways. By the rule of product, $N(c_1) = 1 \cdot P(6,3)$.

For $N(c_2)$ we get 2 ways to do the first step (2 forbidden seats) and again $P(6,3)$ ways for the second, so $N(c_2) = 2 \cdot P(6,3)$.

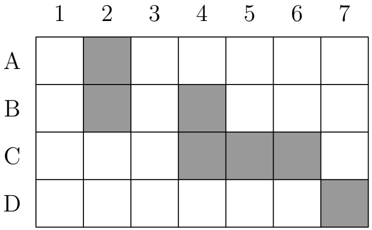
Similarly, $N(c_3) = 3 \cdot P(6,3)$ and $N(c_4) = 1 \cdot P(6,3)$.

So, $S_1 = 1 \cdot P(6,3) + 2 \cdot P(6,3) + 3 \cdot P(6,3) + 1 \cdot P(6,3)$ or $S_1 = 7 \cdot P(6,3)$.

Using the rule of product again, we can get $N(c_1c_2) = 1 \cdot P(5,2)$. The number 1 is the number of ways to seat A and B in forbidden seats and $P(5,2)$ is the number of ways to seat the other 2 in the remaining 5 seats.

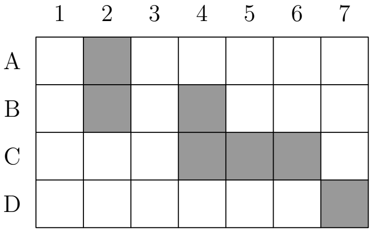


Using the rule of product again, we can get $N(c_1c_2) = 1 \cdot P(5,2)$. The number 1 is the number of ways to seat A and B in forbidden seats and $P(5,2)$ is the number of ways to seat the other 2 in the remaining 5 seats.



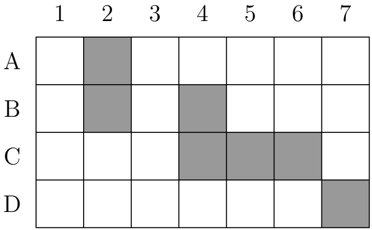
Similarly, $N(c_1c_3) = 3P(5,2)$ and $N(c_2c_3) = 5P(5,2)$. And so on, so that

Using the rule of product again, we can get $N(c_1c_2) = 1 \cdot P(5,2)$. The number 1 is the number of ways to seat A and B in forbidden seats and $P(5,2)$ is the number of ways to seat the other 2 in the remaining 5 seats.



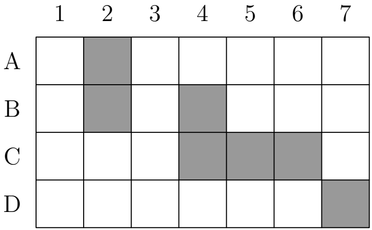
Similarly, $N(c_1c_3) = 3P(5,2)$ and $N(c_2c_3) = 5P(5,2)$. And so on, so that $S_2 = (1+3+5+1+2+3)P(5,2) = 15P(5,2)$.

Using the rule of product again, we can get $N(c_1c_2) = 1 \cdot P(5,2)$. The number 1 is the number of ways to seat A and B in forbidden seats and $P(5,2)$ is the number of ways to seat the other 2 in the remaining 5 seats.



Similarly, $N(c_1c_3) = 3P(5,2)$ and $N(c_2c_3) = 5P(5,2)$. And so on, so that $S_2 = (1+3+5+1+2+3)P(5,2) = 15P(5,2)$. The six numbers 1, 3, 5, 1, 2, and 3 are the number of ways to place 6 different pairs of people in forbidden seats.

Using the rule of product again, we can get $N(c_1c_2) = 1 \cdot P(5,2)$. The number 1 is the number of ways to seat A and B in forbidden seats and $P(5,2)$ is the number of ways to seat the other 2 in the remaining 5 seats.



Similarly, $N(c_1c_3) = 3P(5,2)$ and $N(c_2c_3) = 5P(5,2)$. And so on, so that $S_2 = (1+3+5+1+2+3)P(5,2) = 15P(5,2)$. The six numbers 1, 3, 5, 1, 2, and 3 are the number of ways to place 6 different pairs of people in forbidden seats. Added together they give

$15 =$ the number of ways to place 2 check marks in shaded squares with no two in the same row and no two in the same column

Continuing: $S_3 = 11P(4,1)$, where

$11 =$ the number of ways to place 3 check marks in shaded squares with no two in the same row and no two in the same column, Continuing: $S_3 = 11P(4,1)$, where

$11 =$ the number of ways to place 3 check marks in shaded squares with no two in the same row and no two in the same column,

and $S_4 = 2P(3,0)$, where

$2 =$ the number of ways to place 4 check marks in shaded squares with no two in the same row and no two in the same column,

The rook problem: How many ways are there to place k rooks on a chessboard so that no two are attacking each other?

The rook problem: How many ways are there to place k rooks on a chessboard so that no two are attacking each other?

Restated: How many ways can one place k rooks so that no two are in the same row and no two are in the same column? In a standard 8×8 chessboard, the answer is 0 if $k > 8$.

The rook problem: How many ways are there to place k rooks on a chessboard so that no two are attacking each other?

Restated: How many ways can one place k rooks so that no two are in the same row and no two are in the same column? In a standard 8×8 chessboard, the answer is 0 if $k > 8$.

For $k \leq 8$, the following answers this:

The rook problem: How many ways are there to place k rooks on a chessboard so that no two are attacking each other?

Restated: How many ways can one place k rooks so that no two are in the same row and no two are in the same column? In a standard 8×8 chessboard, the answer is 0 if $k > 8$.

For $k \leq 8$, the following answers this: The k rooks must be on k different rows, so first

1. choose which rows ($C(8, k)$ ways),

The rook problem: How many ways are there to place k rooks on a chessboard so that no two are attacking each other?

Restated: How many ways can one place k rooks so that no two are in the same row and no two are in the same column? In a standard 8×8 chessboard, the answer is 0 if $k > 8$.

For $k \leq 8$, the following answers this: The k rooks must be on k different rows, so first

1. choose which rows ($C(8, k)$ ways), and then
2. assign each rook to a column ($P(8, k)$ ways).

The rook problem: How many ways are there to place k rooks on a chessboard so that no two are attacking each other?

Restated: How many ways can one place k rooks so that no two are in the same row and no two are in the same column? In a standard 8×8 chessboard, the answer is 0 if $k > 8$.

For $k \leq 8$, the following answers this: The k rooks must be on k different rows, so first

1. choose which rows ($C(8, k)$ ways), and then
2. assign each rook to a column ($P(8, k)$ ways).

So there are

$$C(8, k)P(8, k) = \frac{8!}{k!(8-k)!}\frac{8!}{(8-k)!} \quad \text{ways.}$$

The rook problem: How many ways are there to place k rooks on a chessboard so that no two are attacking each other?

Restated: How many ways can one place k rooks so that no two are in the same row and no two are in the same column? In a standard 8×8 chessboard, the answer is 0 if $k > 8$.

For $k \leq 8$, the following answers this: The k rooks must be on k different rows, so first

1. choose which rows ($C(8, k)$ ways), and then
2. assign each rook to a column ($P(8, k)$ ways).

So there are

$$C(8, k)P(8, k) = \frac{8!}{k!(8-k)!}\frac{8!}{(8-k)!} \text{ ways.}$$

Note that if $k = 8$ this equals $1 \cdot 8!$. And if $k = 0$ this equals $1 \cdot 1$.

But what if the chessboard is not the standard size, lets say it has r rows and c columns? The same two-step process gives $C(r, k)P(c, k)$

But what if the chessboard is not the standard size, lets say it has r rows and c columns? The same two-step process gives $C(r, k)P(c, k)$ (or zero if $k > r$ or $k > c$).

But what if the chessboard is not the standard size, lets say it has r rows and c columns? The same two-step process gives $C(r,k)P(c,k)$ (or zero if $k > r$ or $k > c$). (Note: $C(c,k)P(r,k)$ also works.)

But what if the chessboard is not the standard size, lets say it has r rows and c columns? The same two-step process gives $C(r,k)P(c,k)$ (or zero if $k > r$ or $k > c$). (Note: $C(c,k)P(r,k)$ also works.)

And finally, what if the rooks are required to be only in certain marked squares? For example, the shaded squares in our 4×7 seating rectangle.

But what if the chessboard is not the standard size, lets say it has r rows and c columns? The same two-step process gives $C(r,k)P(c,k)$ (or zero if $k > r$ or $k > c$). (Note: $C(c,k)P(r,k)$ also works.)

And finally, what if the rooks are required to be only in certain marked squares? For example, the shaded squares in our 4×7 seating rectangle.

In this last case, the conditions on the rooks (no two in the same row, and no two in the same column) are the same as the conditions on our check marks.

But what if the chessboard is not the standard size, lets say it has r rows and c columns? The same two-step process gives $C(r,k)P(c,k)$ (or zero if $k > r$ or $k > c$). (Note: $C(c,k)P(r,k)$ also works.)

And finally, what if the rooks are required to be only in certain marked squares? For example, the shaded squares in our 4×7 seating rectangle.

In this last case, the conditions on the rooks (no two in the same row, and no two in the same column) are the same as the conditions on our check marks.

We define a *chessboard* to be any collection of squares fitted to a rectangular grid (example picture on the next slide).

But what if the chessboard is not the standard size, lets say it has r rows and c columns? The same two-step process gives $C(r,k)P(c,k)$ (or zero if $k > r$ or $k > c$). (Note: $C(c,k)P(r,k)$ also works.)

And finally, what if the rooks are required to be only in certain marked squares? For example, the shaded squares in our 4×7 seating rectangle.

In this last case, the conditions on the rooks (no two in the same row, and no two in the same column) are the same as the conditions on our check marks.

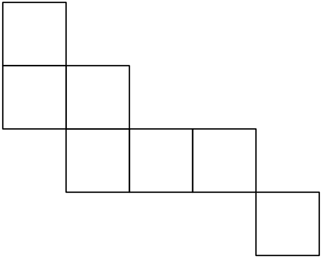
We define a *chessboard* to be any collection of squares fitted to a rectangular grid (example picture on the next slide). We give a chess board a name like C and then define $r_k(C)$ to be the number of ways to place k rooks in C with no two in the same row, and no two in the same column.

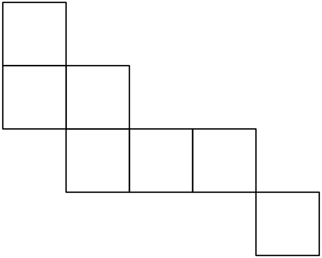
But what if the chessboard is not the standard size, lets say it has r rows and c columns? The same two-step process gives $C(r,k)P(c,k)$ (or zero if $k > r$ or $k > c$). (Note: $C(c,k)P(r,k)$ also works.)

And finally, what if the rooks are required to be only in certain marked squares? For example, the shaded squares in our 4×7 seating rectangle.

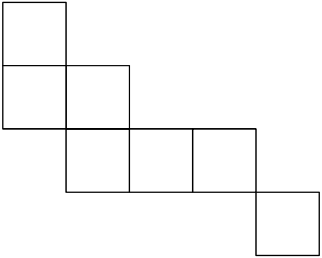
In this last case, the conditions on the rooks (no two in the same row, and no two in the same column) are the same as the conditions on our check marks.

We define a *chessboard* to be any collection of squares fitted to a rectangular grid (example picture on the next slide). We give a chess board a name like C and then define $r_k(C)$ to be the number of ways to place k rooks in C with no two in the same row, and no two in the same column. We call these the *rook numbers for C*. Note that if k is greater than either the number of rows in C or the number of columns then $r_k(C) = 0$.

$C =$ 

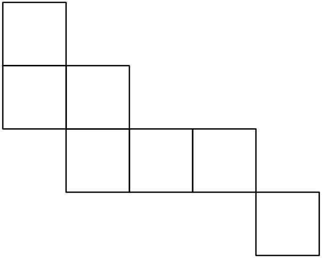
$C =$ 

This C is essentially the shaded squares in our seating problem, except the empty columns have been remove. Removing them has no effect on the number of ways to place check marks (or rooks).

$C =$ 

This C is essentially the shaded squares in our seating problem, except the empty columns have been remove. Removing them has no effect on the number of ways to place check marks (or rooks).

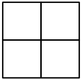
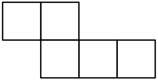
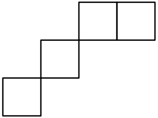
Thus we have seen that $r_1(C) = 7$, $r_2(C) = 15$, $r_3(C) = 11$, and $r_4(C) = 2$. To be complete $r_0(C) = 1$ and $r_k(C) = 0$ for all $k \geq 5$.

$C =$ 

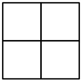
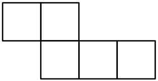
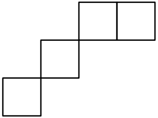
This C is essentially the shaded squares in our seating problem, except the empty columns have been remove. Removing them has no effect on the number of ways to place check marks (or rooks).

Thus we have seen that $r_1(C) = 7$, $r_2(C) = 15$, $r_3(C) = 11$, and $r_4(C) = 2$. To be complete $r_0(C) = 1$ and $r_k(C) = 0$ for all $k \geq 5$. Note that $r_1(C)$ is always the number of squares in C.

Some examples:

C_1 = 
C_2 = 
C_3 = 

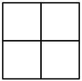
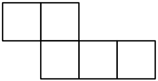
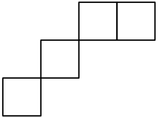
Some examples:

$C_1 =$  $C_2 =$  $C_3 =$ 

Then

1. $r_0 = 1$, $r_1 = 4$, $r_2 = 2$ for C_1.
2. $r_0 = 1$, $r_1 = 5$, $r_2 = 5$ for C_2.
3. $r_0 = 1$, $r_1 = 4$, $r_2 = 5$, $r_3 = 2$ for C_3.

Some examples:

$C_1 =$  $C_2 =$  $C_3 =$ 

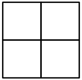
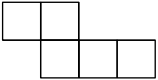
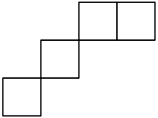
Then

1. $r_0 = 1$, $r_1 = 4$, $r_2 = 2$ for C_1.
2. $r_0 = 1$, $r_1 = 5$, $r_2 = 5$ for C_2.
3. $r_0 = 1$, $r_1 = 4$, $r_2 = 5$, $r_3 = 2$ for C_3.

Rook polynomials. If C is any chessboard, the the rook polynoial for C is

$$r(C, x) = r_0 + r_1 x + r_2 x^2 + \cdots$$

Some examples:

$C_1 =$  $C_2 =$  $C_3 =$ 

Then

1. $r_0 = 1$, $r_1 = 4$, $r_2 = 2$ for C_1.
2. $r_0 = 1$, $r_1 = 5$, $r_2 = 5$ for C_2.
3. $r_0 = 1$, $r_1 = 4$, $r_2 = 5$, $r_3 = 2$ for C_3.

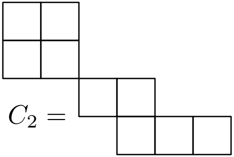
Rook polynomials. If C is any chessboard, the the rook polynoial for C is

$$r(C, x) = r_0 + r_1 x + r_2 x^2 + \cdots$$

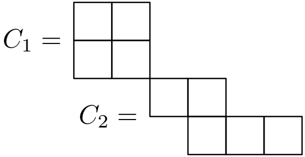
Thus, each power x^k is multiplied by $r_k(C) = r_k$. Since there are only finitely many terms, this is always a polynomial. For example

$$r(C_3, x) = 1 + 4x + 5x^2 + 2x^3$$

Consider this chessboard: $C =$ $C_1 =$ $C_2 =$

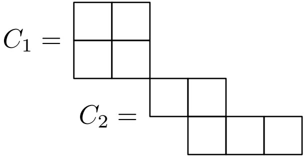


Consider this chessboard: $C =$



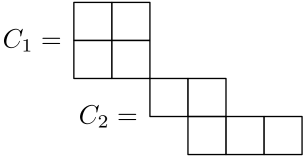
We will be able to produce the rook numbers for C from those for C_1 and C_2.

Consider this chessboard: $C =$



We will be able to produce the rook numbers for C from those for C_1 and C_2. Consider the rook number $r_2(C)$. We break down the possibilities for the 2 rooks into 3 mutually exclusive cases,

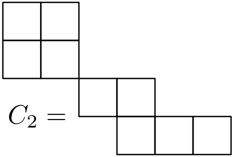
Consider this chessboard: $C =$



We will be able to produce the rook numbers for C from those for C_1 and C_2. Consider the rook number $r_2(C)$. We break down the possibilities for the 2 rooks into 3 mutually exclusive cases,

1. 0 rooks in C_1 and 2 rooks in C_2: $r_0(C_1)r_2(C_2)$ ways.

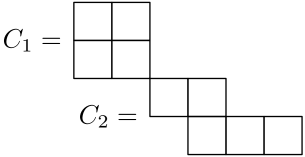
Consider this chessboard: $C =$ $C_1 =$ $C_2 =$



We will be able to produce the rook numbers for C from those for C_1 and C_2. Consider the rook number $r_2(C)$. We break down the possibilities for the 2 rooks into 3 mutually exclusive cases,

1. 0 rooks in C_1 and 2 rooks in C_2: $r_0(C_1)r_2(C_2)$ ways.
2. 1 rook in C_1 and 1 rooks in C_2: $r_1(C_1)r_1(C_2)$ ways.

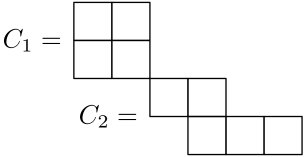
Consider this chessboard: $C =$



We will be able to produce the rook numbers for C from those for C_1 and C_2. Consider the rook number $r_2(C)$. We break down the possibilities for the 2 rooks into 3 mutually exclusive cases,

1. 0 rooks in C_1 and 2 rooks in C_2: $r_0(C_1)r_2(C_2)$ ways.
2. 1 rook in C_1 and 1 rooks in C_2: $r_1(C_1)r_1(C_2)$ ways.
3. 2 rooks in C_1 and 0 rooks in C_2: $r_2(C_1)r_0(C_2)$ ways.

Consider this chessboard: $C =$



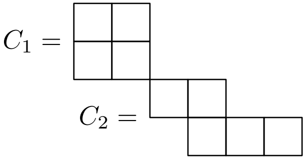
We will be able to produce the rook numbers for C from those for C_1 and C_2. Consider the rook number $r_2(C)$. We break down the possibilities for the 2 rooks into 3 mutually exclusive cases,

1. 0 rooks in C_1 and 2 rooks in C_2: $r_0(C_1)r_2(C_2)$ ways.
2. 1 rook in C_1 and 1 rooks in C_2: $r_1(C_1)r_1(C_2)$ ways.
3. 2 rooks in C_1 and 0 rooks in C_2: $r_2(C_1)r_0(C_2)$ ways.

By the rule of sum,

$$r_2(C) = r_0(C_1)r_2(C_2) + r_1(C_1)r_1(C_2) + r_2(C_1)r_0(C_2).$$

Consider this chessboard: $C =$



We will be able to produce the rook numbers for C from those for C_1 and C_2. Consider the rook number $r_2(C)$. We break down the possibilities for the 2 rooks into 3 mutually exclusive cases,

1. 0 rooks in C_1 and 2 rooks in C_2: $r_0(C_1)r_2(C_2)$ ways.
2. 1 rook in C_1 and 1 rooks in C_2: $r_1(C_1)r_1(C_2)$ ways.
3. 2 rooks in C_1 and 0 rooks in C_2: $r_2(C_1)r_0(C_2)$ ways.

By the rule of sum,

$$r_2(C) = r_0(C_1)r_2(C_2) + r_1(C_1)r_1(C_2) + r_2(C_1)r_0(C_2).$$

This is the formula for the x^2 term in $r(C_1, x)r(C_2, x)$. In fact,

$$\begin{aligned} r(C, x) = r(C_1, x)r(C_2, x) &= (1 + 4x + 2x^2)(1 + 5x + 5x^2) \\ &= 1 + 9x + 27x^2 + 30x^3 + 10x^4 \end{aligned}$$

This product formula is valid only when all the following are satisfied:

This product formula is valid only when all the following are satisfied:

1. C is made up of two disjoint parts C_1 and C_2.

This product formula is valid only when all the following are satisfied:

1. C is made up of two disjoint parts C_1 and C_2.
2. No row in the grid contains squares from both C_1 and C_2.

This product formula is valid only when all the following are satisfied:

1. C is made up of two disjoint parts C_1 and C_2.
2. No row in the grid contains squares from both C_1 and C_2.
3. No column in the grid contains squares from both C_1 and C_2.

This product formula is valid only when all the following are satisfied:

1. C is made up of two disjoint parts C_1 and C_2.
2. No row in the grid contains squares from both C_1 and C_2.
3. No column in the grid contains squares from both C_1 and C_2.

When all 3 of these are satisfied, then $r(C, x) = r(C_1, x)r(C_2, x)$.

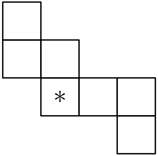
This product formula is valid only when all the following are satisfied:

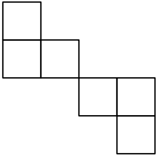
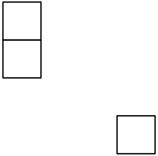
1. C is made up of two disjoint parts C_1 and C_2.
2. No row in the grid contains squares from both C_1 and C_2.
3. No column in the grid contains squares from both C_1 and C_2.

When all 3 of these are satisfied, then $r(C, x) = r(C_1, x)r(C_2, x)$.

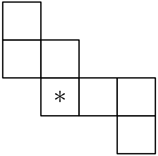
This is useful by itself, when applicable, but we need another tool for computing $r(C, x)$ that is applicable even when this one is not.

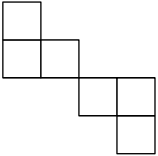
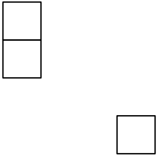
Consider the following chessboards:

$C =$ 

$C_e =$ 
$C_s =$ 

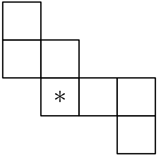
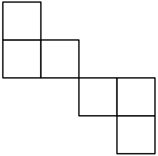
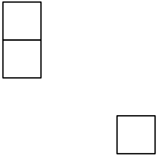
Consider the following chessboards:

$C =$ 

$C_e =$ 
$C_s =$ 

In computing $r_k(C)$ we have two cases:

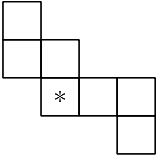
Consider the following chessboards:

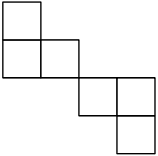
$C =$  $C_e =$  $C_s =$ 

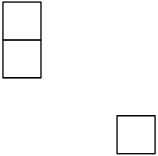
In computing $r_k(C)$ we have two cases:

1. Either there is no rook in the marked square, so all k rooks lie in C_e

Consider the following chessboards:

$C =$ 

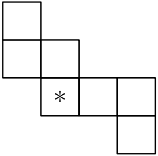
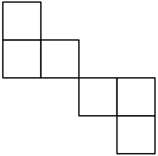
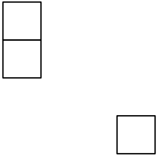
$C_e =$ 

$C_s =$ 

In computing $r_k(C)$ we have two cases:

1. Either there is no rook in the marked square, so all k rooks lie in C_e
2. or there is rook in the marked square and the other $k-1$ rooks lie in C_s.

Consider the following chessboards:

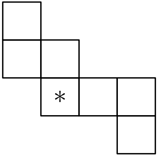
$C =$  $C_e =$  $C_s =$ 

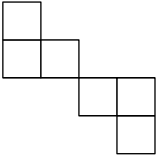
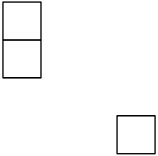
In computing $r_k(C)$ we have two cases:

1. Either there is no rook in the marked square, so all k rooks lie in C_e
2. or there is rook in the marked square and the other $k-1$ rooks lie in C_s.

The first case has $r_k(C_e)$ possibilities and the second case has $r_{k-1}(C_s)$ possibilities.

Consider the following chessboards:

$C =$ 

$C_e =$ 
$C_s =$ 

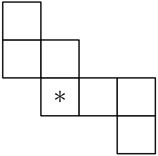
In computing $r_k(C)$ we have two cases:

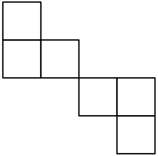
1. Either there is no rook in the marked square, so all k rooks lie in C_e
2. or there is rook in the marked square and the other $k-1$ rooks lie in C_s.

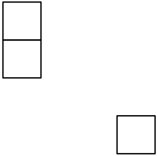
The first case has $r_k(C_e)$ possibilities and the second case has $r_{k-1}(C_s)$ possibilities. Therefore

$$r_k(C) = r_k(C_e) + r_{k-1}(C_s).$$

Consider the following chessboards:

$C =$ 

$C_e =$ 

$C_s =$ 

In computing $r_k(C)$ we have two cases:

1. Either there is no rook in the marked square, so all k rooks lie in C_e
2. or there is rook in the marked square and the other $k-1$ rooks lie in C_s.

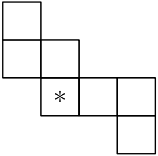
The first case has $r_k(C_e)$ possibilities and the second case has $r_{k-1}(C_s)$ possibilities. Therefore

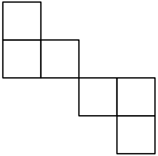
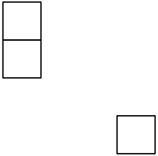
$$r_k(C) = r_k(C_e) + r_{k-1}(C_s).$$

Or, what is the same

$$r_k(C)x^k = r_k(C_e)x^k + x \cdot r_{k-1}(C_s)x^{k-1}.$$

Consider the following chessboards:

$C =$ 

$C_e =$ 
$C_s =$ 

In computing $r_k(C)$ we have two cases:

1. Either there is no rook in the marked square, so all k rooks lie in C_e
2. or there is rook in the marked square and the other $k-1$ rooks lie in C_s.

The first case has $r_k(C_e)$ possibilities and the second case has $r_{k-1}(C_s)$ possibilities. Therefore

$$r_k(C) = r_k(C_e) + r_{k-1}(C_s).$$

Or, what is the same

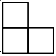
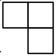
$$r_k(C)x^k = r_k(C_e)x^k + x \cdot r_{k-1}(C_s)x^{k-1}.$$

Adding these we get the formula for $r(C, x)$:

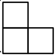
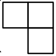
$$r(C, x) = r(C_e, x) + x \cdot r(C_s, x).$$

So, to compute $r(C, x)$ we need $r(C_e, x)$ and $r(C_s, x)$ and these both have the form required for the product formula:

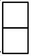
So, to compute $r(C, x)$ we need $r(C_e, x)$ and $r(C_s, x)$ and these both have the form required for the product formula:

$r(C_e, x) = r\Big($  $, x\Big)\, r\Big($  $, x\Big) = (1 + 3x + x^2)^2$

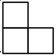
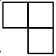
So, to compute $r(C, x)$ we need $r(C_e, x)$ and $r(C_s, x)$ and these both have the form required for the product formula:

$r(C_e, x) = r\Big($  $, x\Big)\, r\Big($  $, x\Big) = (1 + 3x + x^2)^2$

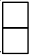
and

$r(C_s, x) = r\Big($  $, x\Big)\, r\big($  $, x\big) = (1 + 2x)(1 + x)$

So, to compute $r(C, x)$ we need $r(C_e, x)$ and $r(C_s, x)$ and these both have the form required for the product formula:

$r(C_e, x) = r\Big($  $, x\Big)\, r\Big($  $, x\Big) = (1 + 3x + x^2)^2$

and

$r(C_s, x) = r\Big($  $, x\Big)\, r\big($  $, x\big) = (1 + 2x)(1 + x)$

So that,

$$\begin{aligned} r(C, x) &= r(C_e, x) + x \cdot r(C_s, x) \\ r(C, x) &= (1 + 3x + x^2)^2 + x(1 + 2x)(1 + x) \\ &= 1 + 7x + 14x^2 + 8x^3 + x^4 \end{aligned}$$