Derangements

Daniel H. Luecking

September 8, 2023

The unexpected incident of e in the denominator

If you've had Calculus II you may remember the formula

$$e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!}$$

The unexpected incident of e in the denominator

If you've had Calculus II you may remember the formula

$$e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!}$$

Putting $x = -1$, we get

$$\begin{aligned} \frac{1}{e} &= \frac{1}{0!} - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} - \cdots \\ &= \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} - \cdots \end{aligned}$$

The unexpected incident of e in the denominator

If you've had Calculus II you may remember the formula

$$e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!}$$

Putting $x = -1$, we get

$$\begin{aligned} \frac{1}{e} &= \frac{1}{0!} - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} - \cdots \\ &= \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} - \cdots \end{aligned}$$

This is is an infinite sum, but we can get a pretty accurate value by stopping the sum after a relative modest number of terms.

The unexpected incident of e in the denominator

If you've had Calculus II you may remember the formula

$$e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!}$$

Putting $x = -1$, we get

$$\begin{aligned} \frac{1}{e} &= \frac{1}{0!} - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} - \cdots \\ &= \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} - \cdots \end{aligned}$$

This is is an infinite sum, but we can get a pretty accurate value by stopping the sum after a relative modest number of terms.

In particular, we get the approximation $d_n \approx n!/e$ and the probability of a derangement is approximately $1/e$. This is accurate to at least 6 decimal places for $n \geq 10$.

Here are some computed values.

- $d_2 = 1$ and the probability of a derangement is 0.5

Here are some computed values.

- $d_2 = 1$ and the probability of a derangement is 0.5
- $d_5 = 44$ and the probability of a derangement is about 0.366...

Here are some computed values.

- $d_2 = 1$ and the probability of a derangement is 0.5
- $d_5 = 44$ and the probability of a derangement is about 0.366...
- $d_8 = 14833$ and the probability of a derangement is about 0.367881944...

Here are some computed values.

- $d_2 = 1$ and the probability of a derangement is 0.5
- $d_5 = 44$ and the probability of a derangement is about 0.366...
- $d_8 = 14833$ and the probability of a derangement is about 0.367881944...

By comparison $1/e = 0.367879441171442321595523770161460\ldots$

Here are some computed values.

- $d_2 = 1$ and the probability of a derangement is 0.5
- $d_5 = 44$ and the probability of a derangement is about 0.366...
- $d_8 = 14833$ and the probability of a derangement is about 0.367881944...

By comparison $1/e = 0.3678794411714423215955237701614\,6\ldots$

Despite the previous discussion, producing the value $10!/e$ for d_{10} will not be accepted as correct, as it is only approximate.

Here are some computed values.

- $d_2 = 1$ and the probability of a derangement is 0.5
- $d_5 = 44$ and the probability of a derangement is about 0.366...
- $d_8 = 14833$ and the probability of a derangement is about 0.367881944...

By comparison $1/e = 0.36787944117144232159552377016146\ldots$

Despite the previous discussion, producing the value $10!/e$ for d_{10} will not be accepted as correct, as it is only approximate.

In the analysis of derangements via inclusion-exclusion, when we have n objects being permuted, we have the formula $S_k = \dfrac{n!}{k!}$.

This simple formula for S_k allows us to quickly obtain the number of permutations that have at least 3 objects in their original place:

$$\begin{aligned} L_3 &= S_3 - \binom{3}{1} S_4 + \binom{4}{2} S_5 - \cdots \pm \binom{n-1}{n-3} S_n \\ &= \frac{n!}{3!} - 3\,\frac{n!}{4!} + 6\,\frac{n!}{5!} - \cdots \pm \frac{(n-1)!}{2!(n-3)!}\frac{n!}{n!} \end{aligned}$$

This simple formula for S_k allows us to quickly obtain the number of permutations that have at least 3 objects in their original place:

$$L_3 = S_3 - \binom{3}{1} S_4 + \binom{4}{2} S_5 - \cdots \pm \binom{n-1}{n-3} S_n$$
$$= \frac{n!}{3!} - 3\,\frac{n!}{4!} + 6\,\frac{n!}{5!} - \cdots \pm \frac{(n-1)!}{2!(n-3)!}\frac{n!}{n!}$$

However there is an easier way to compute E_k. Since E_k is the number of permutations with exactly k objects in their original positions, we can create such permutations in two steps:

1. Pick which k positions to leave the same: $C(n,k)$ ways.
2. Perform a derangement of the remaining $n-k$ objects: d_{n-k} ways.

This simple formula for S_k allows us to quickly obtain the number of permutations that have at least 3 objects in their original place:

$$L_3 = S_3 - \binom{3}{1} S_4 + \binom{4}{2} S_5 - \cdots \pm \binom{n-1}{n-3} S_n$$
$$= \frac{n!}{3!} - 3\,\frac{n!}{4!} + 6\,\frac{n!}{5!} - \cdots \pm \frac{(n-1)!}{2!(n-3)!}\frac{n!}{n!}$$

However there is an easier way to compute E_k. Since E_k is the number of permutations with exactly k objects in their original positions, we can create such permutations in two steps:

1. Pick which k positions to leave the same: $C(n, k)$ ways.
2. Perform a derangement of the remaining $n - k$ objects: d_{n-k} ways.

By the rule of product, $E_k = C(n, k)d_{n-k} = \dfrac{n!}{k!(n-k)!}d_{n-k}$.

This simple formula for S_k allows us to quickly obtain the number of permutations that have at least 3 objects in their original place:

$$\begin{aligned} L_3 &= S_3 - \binom{3}{1} S_4 + \binom{4}{2} S_5 - \cdots \pm \binom{n-1}{n-3} S_n \\ &= \frac{n!}{3!} - 3\,\frac{n!}{4!} + 6\,\frac{n!}{5!} - \cdots \pm \frac{(n-1)!}{2!(n-3)!}\frac{n!}{n!} \end{aligned}$$

However there is an easier way to compute E_k. Since E_k is the number of permutations with exactly k objects in their original positions, we can create such permutations in two steps:

1. Pick which k positions to leave the same: $C(n,k)$ ways.
2. Perform a derangement of the remaining $n-k$ objects: d_{n-k} ways.

By the rule of product, $E_k = C(n,k)d_{n-k} = \dfrac{n!}{k!(n-k)!}d_{n-k}$. Compare that to the formula

$$E_k = \frac{n!}{k!} - \binom{k+1}{1}\frac{n!}{(k+1)!} + \binom{k+2}{2}\frac{n!}{(k+2)!} - \cdots \pm \binom{n}{n-k}\frac{n!}{n!}.$$

Examples

Suppose 7 students are asked to deposit their smartphones in a box before taking a test. Then the phones are returned randomly after the test.

Examples

Suppose 7 students are asked to deposit their smartphones in a box before taking a test. Then the phones are returned randomly after the test.

How many ways can they all get the wrong phone?

Examples

Suppose 7 students are asked to deposit their smartphones in a box before taking a test. Then the phones are returned randomly after the test.

How many ways can they all get the wrong phone? This is a derangement of the original function associating owner to phone. There are d_7 of those and

$$d_7 = 7!\left(\frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} - \frac{1}{5!} + \frac{1}{6!} - \frac{1}{7!}\right)$$

is one way to write the value.

Examples

Suppose 7 students are asked to deposit their smartphones in a box before taking a test. Then the phones are returned randomly after the test.

How many ways can they all get the wrong phone? This is a derangement of the original function associating owner to phone. There are d_7 of those and

$$d_7 = 7!\left(\frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} - \frac{1}{5!} + \frac{1}{6!} - \frac{1}{7!}\right)$$

is one way to write the value.

How many ways can at least one student get the right phone?

Examples

Suppose 7 students are asked to deposit their smartphones in a box before taking a test. Then the phones are returned randomly after the test.

How many ways can they all get the wrong phone? This is a derangement of the original function associating owner to phone. There are d_7 of those and

$$d_7 = 7!\left(\frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} - \frac{1}{5!} + \frac{1}{6!} - \frac{1}{7!}\right)$$

is one way to write the value.

How many ways can at least one student get the right phone?
This is every permutation except the derangements and there are $7! - d_7$ of these.

How many ways can exactly 3 students get their own phone?

Examples

Suppose 7 students are asked to deposit their smartphones in a box before taking a test. Then the phones are returned randomly after the test.

How many ways can they all get the wrong phone? This is a derangement of the original function associating owner to phone. There are d_7 of those and

$$d_7 = 7!\left(\frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} - \frac{1}{5!} + \frac{1}{6!} - \frac{1}{7!}\right)$$

is one way to write the value.

How many ways can at least one student get the right phone?
This is every permutation except the derangements and there are $7! - d_7$ of these.

How many ways can exactly 3 students get their own phone?
This is $E_3 = C(7,3)d_4 = \dfrac{7!}{3!4!}\,4!\left(\dfrac{1}{2!} - \dfrac{1}{3!} + \dfrac{1}{4!}\right)$.

An engineer is confronted with 6 cables each of which needs to be plugged into a specific socket. The engineer has forgotten to label the cables and sockets. If they are plugged in randomly, how many ways could they all be wrong?

An engineer is confronted with 6 cables each of which needs to be plugged into a specific socket. The engineer has forgotten to label the cables and sockets. If they are plugged in randomly, how many ways could they all be wrong?

This is a derangement so the number of ways is d_6. One way to write this is

$$\begin{aligned} d_6 &= \frac{6!}{2!} - \frac{6!}{3!} + \frac{6!}{4!} - \frac{6!}{5!} + \frac{6!}{6!} \\ &= 6 \cdot 5 \cdot 4 \cdot 3 - 6 \cdot 5 \cdot 4 + 6 \cdot 5 - 6 + 1 \\ &= 6(5(4(3-1)+1)-1)+1 \end{aligned}$$

An engineer is confronted with 6 cables each of which needs to be plugged into a specific socket. The engineer has forgotten to label the cables and sockets. If they are plugged in randomly, how many ways could they all be wrong?

This is a derangement so the number of ways is d_6. One way to write this is

$$\begin{aligned} d_6 &= \frac{6!}{2!} - \frac{6!}{3!} + \frac{6!}{4!} - \frac{6!}{5!} + \frac{6!}{6!} \\ &= 6 \cdot 5 \cdot 4 \cdot 3 - 6 \cdot 5 \cdot 4 + 6 \cdot 5 - 6 + 1 \\ &= 6(5(4(3-1)+1)-1)+1 \end{aligned}$$

I wrote the last line to illustrate the most efficient way to calculate d_6. In general, if you need to calculate d_n efficiently, a recursive method based on $d_n = nd_{n-1} + (-1)^n$ is probably best. For example $d_2 = 1$, so $d_3 = 3 \cdot d_2 - 1 = 2$, $d_4 = 4 \cdot d_3 + 1 = 9$, $d_5 = 5 \cdot d_4 - 1 = 44$, and so on.

How many ways could exactly half of them be wrong?

How many ways could exactly half of them be wrong? This means 3 are correct: $E_3 = C(6,3)d_3 = \frac{6!}{3!3!} \cdot \left(\frac{3!}{2!} - \frac{3!}{3!}\right) = 40.$

How many ways could exactly half of them be wrong? This means 3 are correct: $E_3 = C(6,3)d_3 = \frac{6!}{3!3!} \cdot \left(\frac{3!}{2!} - \frac{3!}{3!}\right) = 40.$

How many ways could at most half of them be wrong? This means at least 3 are correct so

$L_3 = \frac{6!}{3!} - \binom{3}{1}\frac{6!}{4!} + \binom{4}{2}\frac{6!}{5!} - \binom{5}{3}\frac{6!}{6!} = 120 - 90 + 36 - 10 = 56.$

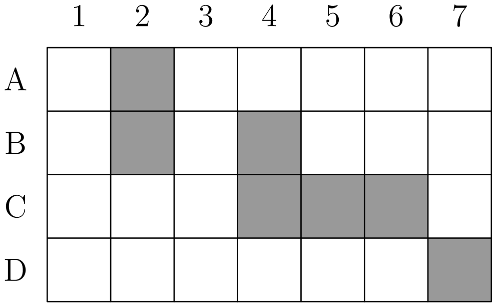
Selections with Forbidden Choices

It is not hard to determine how many ways 4 people can be placed in 7 seats: Pick 4 seats and then assign each to one of the persons. This is a permutation and so there are $P(7,4) = 7!/(7-4)!$ ways.

Selections with Forbidden Choices

It is not hard to determine how many ways 4 people can be placed in 7 seats: Pick 4 seats and then assign each to one of the persons. This is a permutation and so there are $P(7,4) = 7!/(7-4)!$ ways.

But suppose some persons refuse to sit in some seats? Here is a visual representation where a shaded square indicates that the person on the left refuses to sit in the seat above.



Seat assignments with forbidden choices

We pose the problem: how many ways can the 4 people be seated with no one seated in a forbidden seat?

Seat assignments with forbidden choices

We pose the problem: how many ways can the 4 people be seated with no one seated in a forbidden seat?

We can visualize an assignment of seats as a check mark placed in each row in the square underneath the seat assigned to that person.

Seat assignments with forbidden choices

We pose the problem: how many ways can the 4 people be seated with no one seated in a forbidden seat?

We can visualize an assignment of seats as a check mark placed in each row in the square underneath the seat assigned to that person. The requirements for these marks is that there is only one in each row and at most one in each column,

Seat assignments with forbidden choices

We pose the problem: how many ways can the 4 people be seated with no one seated in a forbidden seat?

We can visualize an assignment of seats as a check mark placed in each row in the square underneath the seat assigned to that person. The requirements for these marks is that there is only one in each row and at most one in each column, plus the requirement that none is in a shaded square.

Seat assignments with forbidden choices

We pose the problem: how many ways can the 4 people be seated with no one seated in a forbidden seat?

We can visualize an assignment of seats as a check mark placed in each row in the square underneath the seat assigned to that person. The requirements for these marks is that there is only one in each row and at most one in each column, plus the requirement that none is in a shaded square.

We will ultimately use inclusion-exclusion to solve this.

Seat assignments with forbidden choices

We pose the problem: how many ways can the 4 people be seated with no one seated in a forbidden seat?

We can visualize an assignment of seats as a check mark placed in each row in the square underneath the seat assigned to that person. The requirements for these marks is that there is only one in each row and at most one in each column, plus the requirement that none is in a shaded square.

We will ultimately use inclusion-exclusion to solve this. The conditions are

1. c_1: The check mark in the first row is in a shaded square.
2. c_2: The check mark in the second row is in a shaded square.
3. c_3: The check mark in the third row is in a shaded square.
4. c_4: The check mark in the fourth row is in a shaded square.

Seat assignments with forbidden choices

We pose the problem: how many ways can the 4 people be seated with no one seated in a forbidden seat?

We can visualize an assignment of seats as a check mark placed in each row in the square underneath the seat assigned to that person. The requirements for these marks is that there is only one in each row and at most one in each column, plus the requirement that none is in a shaded square.

We will ultimately use inclusion-exclusion to solve this. The conditions are

1. c_1: The check mark in the first row is in a shaded square.
2. c_2: The check mark in the second row is in a shaded square.
3. c_3: The check mark in the third row is in a shaded square.
4. c_4: The check mark in the fourth row is in a shaded square.

What we want is $N(\overline{c_1}\,\overline{c_2}\,\overline{c_3}\,\overline{c_4}) = N - S_1 + S_2 - S_3 + S_4$.

Seat assignments with forbidden choices

We pose the problem: how many ways can the 4 people be seated with no one seated in a forbidden seat?

We can visualize an assignment of seats as a check mark placed in each row in the square underneath the seat assigned to that person. The requirements for these marks is that there is only one in each row and at most one in each column, plus the requirement that none is in a shaded square.

We will ultimately use inclusion-exclusion to solve this. The conditions are

1. c_1: The check mark in the first row is in a shaded square.
2. c_2: The check mark in the second row is in a shaded square.
3. c_3: The check mark in the third row is in a shaded square.
4. c_4: The check mark in the fourth row is in a shaded square.

What we want is $N(\overline{c_1}\,\overline{c_2}\,\overline{c_3}\,\overline{c_4}) = N - S_1 + S_2 - S_3 + S_4$. It turns out that finding N and S_1 are not hard, finding other S_k takes some work.