Extensions of Inclusion-Exclusion

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September 6, 2023

Inclusion-Exclusion

For m conditions $c_1, c_2, \ldots, c_m$, we let N be the number of object to which these conditions apply, and we define S_1 through S_m by

1. $S_1 = N(c_1) + N(c_2) + N(c_3) + \cdots + N(c_m)$.
2. $S_2 = N(c_1c_2) + N(c_1c_3) + N(c_2c_3) + \cdots + N(c_{m-1}c_m)$.
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The formula for the number that satisfy at least one condition is

$$S_1 - S_2 + S_3 - \cdots \pm S_m$$

and the number that satisfy none of the conditions is

$$N - S_1 + S_2 - S_3 + \cdots \mp S_m$$

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Another easy one is $L_0 = N$. We also have
$E_m = L_m = S_m = N(c_1c_2 \ldots c_m)$ and if $k < m$, $E_k = L_k - L_{k+1}$.

To illustrate some of the issues involved, if there are 3 conditions we have

$$E_1 = N(c_1\overline{c_2}\,\overline{c_3}) + N(\overline{c_1}c_2\overline{c_3}) + N(\overline{c_1}\,\overline{c_2}c_3)$$

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This extends to any number of conditions

$$E_1 = S_1 - 2S_2 + 3S_3 - 4S_4 + \cdots \pm mS_m$$

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If there are m conditions and $0 \le k \le m$ then

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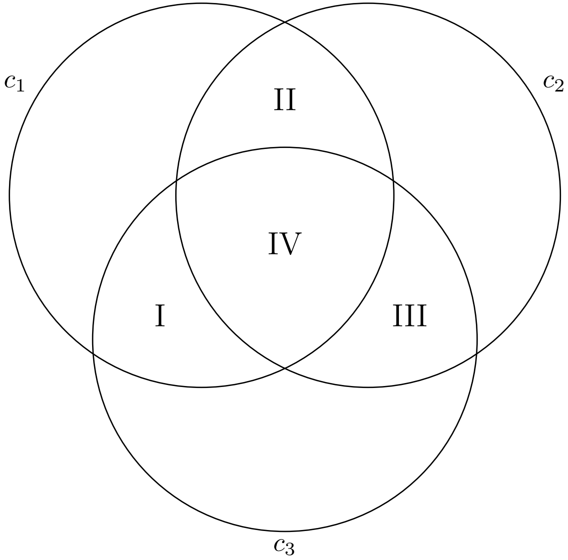

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Solution: As before we let $c_1 =$ 'contains "DO"' $c_2 =$ 'contains "RE"' $c_3 =$ 'contains "MI"'.

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(a) $E_1 = S_1 - 2S_2 + 3S_3 = 3 \cdot 8! - 2 \cdot 3 \cdot 7! + 3 \cdot 6! = 92{,}880.$

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(c) $L_2 = S_2 - 2S_3 = 3 \cdot 7! - 2 \cdot 6! = 13{,}680.$

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Returning to the arrangements of "BOOKBINDING" with conditions $c_1 =$ 'contains "BB"', $c_2 =$ 'contains "OO"', $c_3 =$ 'contains "II"', and $c_4 =$ 'contains "NN"'.

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(a) How many contain exactly 2 of those substrings? Looking up

$$E_2 = S_2 - \binom{3}{1}S_3 + \binom{4}{2}S_4 = S_2 - 3S_3 + 6S_4 = 6\,\frac{9!}{2!2!} - 3 \cdot 4\,\frac{8!}{2!} + 6 \cdot 7!$$

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(a) How many contain exactly 2 of those substrings? Looking up $E_2 = S_2 - \binom{3}{1}S_3 + \binom{4}{2}S_4 - \binom{5}{3}S_5 = S_2 - 3S_3 + 6S_4 - 10S_5 = 10\,\frac{10!}{2!2!2!} - 3 \cdot 10\,\frac{9!}{2!2!} + 6 \cdot 5\frac{8!}{2!} - 10 \cdot 7!$.

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(c) At least 3?

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If we have a given permutation of all the elements of a set, then the permutations that are different from it in every position are called *derangements* of that permutation.

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If A is the set of men and B is the set of hats we have the original function assigning to each man his own hat. Afterwards we have the new function that assigns to each man the hat he is handed. To compute the probability we need to divide the number of derangements by the number of all one-to-one functions.

We attack the problem with inclusion exclusion. Let's number the men from 1 to 15 and let $c_j =$ 'man number j gets his own hat'. Note that N is the total number of permutations so $N = 15!$.

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By a similar argument, for any two conditions $N(c_j c_k) = 13!$. If we take the sum of all these (there are $C(15, 2)$ terms) we get $S_2 = \binom{15}{2} 13! = \dfrac{15!}{2!13!} 13! = \dfrac{15!}{2!}$.

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$$\begin{aligned}
N(\overline{c_1}\,\overline{c_2}\ldots\overline{c_{15}}) &= N - S_1 + S_2 - S_3 + S_4 - \cdots - S_{15} \\
&= 15! - 15! + \frac{15!}{2!} - \frac{15!}{3!} + \frac{15!}{4!} - \cdots - \frac{15!}{15!} \\
&= 15!\left(\frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} - \cdots - \frac{1}{15!}\right)
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So the probability of a derangement is $\dfrac{1}{2!} - \dfrac{1}{3!} + \cdots - \dfrac{1}{15!}$ which is approximately 0.36788.

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The general formula for the number of derangements of a permutation with length n is

$$d_n = \frac{n!}{2!} - \frac{n!}{3!} + \frac{n!}{4!} - \cdots \pm \frac{n!}{n!}$$