Basic Principals of Combinatorics

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August 23, 2023

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This is because A and B are disjoint and $C = A \cup B$, so $|C| = |A| + |B|$.

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Now, how many solutions does $x + y + z = 10$ have? We can break this into 11 cases:
Case $z = 0$: so $x + y = 10$ and there are 11 ways.
Case $z = 1$: so $x + y = 9$ and there are 10 ways.
...
Case $z = 10$: so $x + y = 0$ and there is 1 way.

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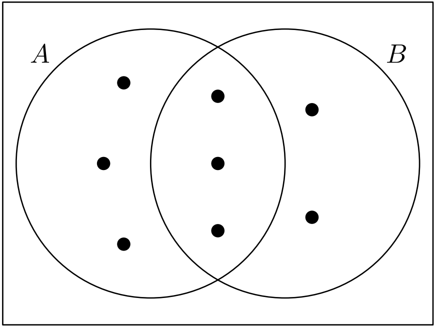
2. If we use this to count final outcomes, we must have
   - If any one task is done differently, the final outcome is different.
   - All final outcomes are produced by some way of performing the sequence of tasks.

A picture of one-to-one correspondence

$$\begin{array}{ccc} 1 & \longleftrightarrow & A \\ 2 & \longleftrightarrow & B \\ 3 & \longleftrightarrow & C \\ 4 & \longleftrightarrow & D \\ \vdots & \vdots & \vdots \\ 24 & \longleftrightarrow & X \\ 25 & \longleftrightarrow & Y \\ 26 & \longleftrightarrow & Z \end{array}$$

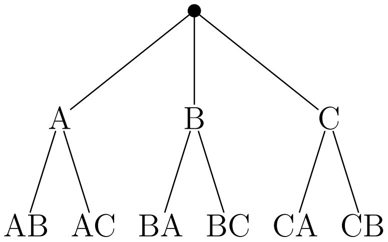
A picture of the rule of sum



$|A| = 6$, $|B| = 5$, $|A \cap B| = 3$, $|A \cup B| = 6 + 5 - 3 = 8$.

A picture of the rule of product

The following illustrates building strings of length 2 from the letters A, B, and C without repeated letters. This illustrates the rule of product: the first task (select a letter) produces a 3-way branching, and then the second task (select a different letter) produces 2-way branching. There are a total of $3 \cdot 2$ paths to the different final results.



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If the set we are choosing from is an n-set and we are choosing k elements, we refer to these as *permutations of n things taken k at a time*. (The shorter '*k-permutations of an n-set*' is also used.)

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The Rule of product comes to our aid here: we can build a k-permutation in k steps: Pick the element for each position one-by-one.

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The special case $P(n,n)$ is important: $P(n,n)=n(n-1)(n-2)\cdots 1$. We have a special notation for this product, called a factorial:

$$n! = n(n-1)(n-2)\cdots 1 \quad \text{(special case: } 0! = 1\text{)}$$

With this notation we have the shorter formulas:

$$P(n,n)=n! \quad \text{and} \quad P(n,k)=\frac{n!}{(n-k)!}.$$

Other ways to look at permutations

Note that $P(100, 3) = 100!/97! = 100 \cdot 99 \cdot 98 = 970200$. The first computation (dividing $100!$ by $97!$) is impossible to do by hand, and not even guaranteed to be exact in some computer programs, the second is easy and pretty quick.

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Consider the set $S = \{A, B, C, D, E, F, G\}$ and the following table:

x	1	2	3	4
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In fact, there is a one-to-one correspondence between permutations and one-to-one functions. That is, for any k-set A and n-set S, $P(n, k)$ is the number of one-to-one functions from A to S.

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If we group all the permutations into clusters of size 6 that each represent the same subset, we get $P(10, 3)/P(3, 3) = \frac{10 \cdot 9 \cdot 8}{3 \cdot 2 \cdot 1} = 120$ subsets.

Permutations with repetition (arrangements)

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To be clear what this means, imagine we write each letter on a tile, $\boxed{\texttt{B}}$ $\boxed{\texttt{O}}$ $\boxed{\texttt{O}}$ $\boxed{\texttt{K}}$ $\boxed{\texttt{K}}$ $\boxed{\texttt{E}}$ $\boxed{\texttt{E}}$ $\boxed{\texttt{P}}$ $\boxed{\texttt{E}}$ $\boxed{\texttt{R}}$, then we shuffle these tiles and line them up, *recording the resulting string* of letters.

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Since there are $2!$ ways to permute the $\boxed{\texttt{O}}$s, $2!$ ways to permute the $\boxed{\texttt{K}}$s and $3!$ ways to permute the $\boxed{\texttt{E}}$s, we have $2!2!3!$ different permutations (of the tiles) that produce the same string.

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Other examples

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Finally, what about "STRATIFICATIONAL" and "GASTROENTEROLOGIST"? Give it a try, the hard part is not missing any repetitions.

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How about permutations of the 26 letters of the alphabet contain the substrings "CAT", "DOG" and "KID"? Answer: $20!$, the number of permutations of $\boxed{\texttt{C A T}}$, $\boxed{\texttt{K I D O G}}$, and 18 individual letters.

Selections without order

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Definition

$C(n, k)$ stands for the number of combinations possible when choosing k elements from a set of size n.

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Example: I have 7 books I haven't read and I want to take 3 of them on vacation. There are

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$$C(7, 2)C(5, 2) = \frac{7!}{5!\,2!}\,\frac{5!}{3!2!} = \frac{7 \cdot 6}{2 \cdot 1}\,\frac{5 \cdot 4}{2 \cdot 1} = 210 \text{ ways to do this.}$$