Math 3103 Combinatorics (Luecking) NAME:______________________________
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Second Exam A (solutions) March 27, 2024

1. For each of the following first-order recurrence relations, find the solution that satisfies the given initial condition.

   (a) $a_n = 6a_{n-1}\,,\ \ n \geq 1\,,$
   $a_0 = 3\,.$

**Ans:** Geometric progression: $a_n = (3)6^n$

   (b) $a_n = a_{n-1} + 7\,,\ \ n \geq 1\,,$
   $a_0 = 4\,.$

**Ans:** Arithmetic progression: $a_n = 4 + 7n$

   (c) $a_n = (n^4 + 1)a_{n-1}\,,\ \ n \geq 1\,,$
   $a_0 = 5\,.$

**Ans:** Successive multiplications: $a_n = 5(1^4 + 1)(2^4 + 1)\cdots(n^4 + 1)$.

   (d) $a_n = a_{n-1} + n^2 7^n\,,\ \ n \geq 1\,,$
   $a_0 = 6\,.$

**Ans:** Successive additions: $a_n = 6 + (1^2)7^1 + (2^2)7^2 + \cdots + n^2 7^n$

2. For the following second-order recurrence relation, find (i) the characteristic equation and its roots, (ii) the general solution, and (iii) the solution that satisfies the initial conditions.

   $a_n - 8a_{n-1} + 16a_{n-2} = 0\,,\ \ n \geq 2\,,$
   $a_0 = 1$ and $a_1 = 5,.$

**Ans:** (i) $r^2 - 8r + 16 = 0$ has roots 4 and 4 (repeated root).

(ii) General solution: $C_1 4^n + C_2 n 4^n$.

(iii) The ICs, $C_1 = 1$ and $4C_1 + 4C_2 = 5$, give $C_1 = 1$ and $C_2 = 1/4$. Thus:

$$a_n = 4^n + \frac{1}{4} n 4^n$$

3. For the following second-order recurrence relation, find (i) the characteristic equation and its roots, (ii) the general solution, and (iii) the solution that satisfies the initial conditions.

$$a_n - 4a_{n-1} + 13a_{n-2} = 0\,, \quad n \geq 2\,,$$
$$a_0 = 0 \quad \text{and} \quad a_1 = 4.$$

**Ans:** (i) $r^2 - 4r + 13 = 0$ has roots $r = 2 \pm 3i$.

(ii) General solution: $C_1(2+3i)^n + C_2(2-3i)^n$.

(iii) The ICs, $C_1 + C_2 = 0$ and $C_1(2+3i) + C_2(2-3i) = 6$, give $C_1 = \dfrac{2}{3i}$ and $C_2 = \dfrac{-2}{3i}$. Thus: $a_n = \dfrac{2}{3i}(2+3i)^n - \dfrac{2}{3i}(2-3i)^n$.

4. For the following nonhomogeneous recurrence relation with initial conditions find (i) the homogeneous solution $a_n^{(h)}$, (ii) a particular solution $a_n^{(p)}$, and (iii) the solution that satisfies the initial conditions.

$$a_n - 5a_{n-1} + 6a_{n-2} = 6\,, \quad n \geq 2\,,$$
$$a_0 = 0 \quad \text{and} \quad a_1 = 0\,.$$

**Ans:** (i) The homogeneous solution is $a_n^{(h)} = C_1 2^n + C_2 3^n$.

(ii) A particular solution exists in the form $a_n^{(p)} = A$ for some constant $A$. The recurrence relation gives $A - 5A + 6A = 6$, so $a_n^{(p)} = A = 3$.

(iii) The general solution is therefore $C_1 2^n + C_2 3^n + 3$. The ICs, $C_1 + C_2 + 3 = 0$ and $2C_1 + 3C_2 + 3 = 0$, give $C_1 = -6$ and $C_2 = 3$. Thus:

$$a_n = (-6)2^n + (3)3^n + 3$$

5. For the following recurrence relation problem, find the generating function $F(x)$ for the sequence $a_n$ without actually finding $a_n$.

$a_n - 2a_{n-1} - 4a_{n-2} = 5\,, \quad n \geq 2\,.$
$a_0 = 1, \quad a_1 = 2.$

**Ans:** Multiplying by $x^n$ and summing gives

$$\sum_{n=2}^{\infty} a_n x^n - 2\sum_{n=2}^{\infty} a_{n-1} x^n - 4\sum_{n=2}^{\infty} a_{n-2} x^n = 5\sum_{n=2}^{\infty} x^n$$

This gives us

$$F(x) - 1 - 2x - 2x(F(x) - 1) - 4x^2 F(x) = \frac{5x^2}{1-x}.$$

or

$$(1 - 2x - 4x^2)F(x) = 1 + \frac{5x^2}{1-x} \quad \text{whence} \quad F(x) = \frac{1 + \frac{5x^2}{1-x}}{1 - 2x - 4x^2}$$

6. Do the following for the given rings.

(a) The factorization of 2684 into primes is $2684 = 2^2 \cdot 11 \cdot 61$. Find (i) the number of units and (ii) the number of proper zero divisors in the ring $\mathbb{Z}_{2684}$. Your final answers must be **completely** simplified.

**Ans:** Units: $\phi(2684) = 2^2 \cdot 11 \cdot 61 \left(\frac{1}{2}\right)\left(\frac{10}{11}\right)\left(\frac{60}{61}\right) = 1200$

Proper zero divisors: $2684 - 1200 - 1 = 1483$

(b) In the ring $\mathbb{Z}_{2003}$ find $(100)^{-1}$. Your final answer must be an explicit element of $\mathbb{Z}_{2003}$. (Note: this is **not** the same ring as the one in part (a).)

**Ans:** The Euclidean algorithm gives:

$$\begin{cases} 2003 = 20(100) + 3 \\ 100 = 33(3) + 1 \end{cases} \quad \text{or} \quad \begin{cases} n = 20k + r_1 \\ k = 33r_1 + r_2 \end{cases}$$

where $n = 2003$, $k = 100$, $r_1 = 3$, and $r_2 = 1$. Putting $r_1 = n - 20k$ (from the top equation) into the bottom one gives
$k = 33(n - 20k) + r_2 = 33n - 660k + r_2$ and solving for $r_2$: $r_2 = 661k - 33n$. If we put the values back in, this says $1 = 661 \cdot 100$ in the ring $\mathbb{Z}_{2003}$. Thus $(100)^{-1} = 661$.

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Second Exam B (solutions) March 27, 2024

1. For each of the following first-order recurrence relations, find the solution that satisfies the given initial condition.

   (a) $a_n = 7a_{n-1}\,,\ \ n \geq 1\,,$
   $a_0 = 3\,.$

**Ans:** Geometric progression: $a_n = (3)7^n$

   (b) $a_n = a_{n-1} + 8\,,\ \ n \geq 1\,,$
   $a_0 = 4\,.$

**Ans:** Arithmetic progression: $a_n = 4 + 8n$

   (c) $a_n = (n^5 + 1)a_{n-1}\,,\ \ n \geq 1\,,$
   $a_0 = 5\,.$

**Ans:** Successive multiplications: $a_n = 5(1^5 + 1)(2^5 + 1) \cdots (n^5 + 1)$.

   (d) $a_n = a_{n-1} + n^2 8^n\,,\ \ n \geq 1\,,$
   $a_0 = 6\,.$

**Ans:** Successive additions: $a_n = 6 + (1^2)8^1 + (2^2)8^2 + \cdots + n^2 8^n$

2. For the following second-order recurrence relation, find (i) the characteristic equation and its roots, (ii) the general solution, and (iii) the solution that satisfies the initial conditions.

   $a_n - 8a_{n-1} + 16a_{n-2} = 0\,,\ \ n \geq 2\,,$
   $a_0 = 1$ and $a_1 = 7,.$

**Ans:** (i) $r^2 - 8r + 16 = 0$ has roots 4 and 4 (repeated root).

(ii) General solution: $C_1 4^n + C_2 n 4^n$.

(iii) The ICs, $C_1 = 1$ and $4C_1 + 4C_2 = 7$, give $C_1 = 1$ and $C_2 = 3/4$. Thus:

$$a_n = 4^n + \frac{3}{4} n 4^n$$

3. For the following second-order recurrence relation, find (i) the characteristic equation and its roots, (ii) the general solution, and (iii) the solution that satisfies the initial conditions.

$$a_n - 6a_{n-1} + 13a_{n-2} = 0\,, \quad n \geq 2\,,$$
$$a_0 = 0 \quad \text{and} \quad a_1 = 6.$$

**Ans:** (i) $r^2 - 6r + 13 = 0$ has roots $r = 3 \pm 2i$.

(ii) General solution: $C_1(3+2i)^n + C_2(3-2i)^n$.

(iii) The ICs, $C_1 + C_2 = 0$ and $C_1(3+2i) + C_2(3-2i) = 4$, give $C_1 = \dfrac{3}{2i}$ and $C_2 = \dfrac{-3}{2i}$. Thus: $a_n = \dfrac{3}{2i}(3+2i)^n - \dfrac{3}{2i}(3-2i)^n$.

4. For the following nonhomogeneous recurrence relation with initial conditions find (i) the homogeneous solution $a_n^{(h)}$, (ii) a particular solution $a_n^{(p)}$, and (iii) the solution that satisfies the initial conditions.

$$a_n - 6a_{n-1} + 8a_{n-2} = 6\,, \quad n \geq 2\,,$$
$$a_0 = 0 \quad \text{and} \quad a_1 = 0\,.$$

**Ans:** (i) The homogeneous solution is $a_n^{(h)} = C_1 2^n + C_2 4^n$.

(ii) A particular solution exists in the form $a_n^{(p)} = A$ for some constant $A$. The recurrence relation gives $A - 6A + 8A = 6$, so $a_n^{(p)} = A = 2$.

(iii) The general solution is therefore $C_1 2^n + C_2 4^n + 2$. The ICs, $C_1 + C_2 + 2 = 0$ and $2C_1 + 4C_2 + 2 = 0$, give $C_1 = -3$ and $C_2 = 1$. Thus:

$$a_n = (-3)2^n + (1)4^n + 2$$

5. For the following recurrence relation problem, find the generating function $F(x)$ for the sequence $a_n$ without actually finding $a_n$.

$a_n - 2a_{n-1} - 4a_{n-2} = 5\,, \quad n \geq 2\,.$
$a_0 = 2, \quad a_1 = 4.$

**Ans:** Multiplying by $x^n$ and summing gives

$$\sum_{n=2}^{\infty} a_n x^n - 2\sum_{n=2}^{\infty} a_{n-1} x^n - 4\sum_{n=2}^{\infty} a_{n-2} x^n = 5\sum_{n=2}^{\infty} x^n$$

This gives us

$$F(x) - 2 - 4x - 2x(F(x) - 2) - 4x^2 F(x) = \frac{5x^2}{1-x}.$$

or

$$(1 - 2x - 4x^2)F(x) = 2 + \frac{5x^2}{1-x} \quad \text{whence} \quad F(x) = \frac{2 + \frac{5x^2}{1-x}}{1 - 2x - 4x^2}$$

6. Do the following for the given rings.

(a) The factorization of 3069 into primes is $3069 = 3^2 \cdot 11 \cdot 31$. Find (i) the number of units and (ii) the number of proper zero divisors in the ring $\mathbb{Z}_{3069}$. Your final answers must be **completely** simplified.

**Ans:** Units: $\phi(3069) = 3^2 \cdot 11 \cdot 31 \left(\frac{2}{3}\right)\left(\frac{10}{11}\right)\left(\frac{30}{31}\right) = 1800$

Proper zero divisors: $3069 - 1800 - 1 = 1268$

(b) In the ring $\mathbb{Z}_{2009}$ find $(100)^{-1}$. Your final answer must be an explicit element of $\mathbb{Z}_{2009}$. (Note: this is **not** the same ring as the one in part (a).)

**Ans:** The Euclidean algorithm gives:

$$\begin{cases} 2009 = 20(100) + 9 \\ \;100 = \;\;\;11(9) + 1 \end{cases} \quad \text{or} \quad \begin{cases} n = 20k \;\; + r_1 \\ k = 11r_1 + r_2 \end{cases}$$

where $n = 2009$, $k = 100$, $r_1 = 9$, and $r_2 = 1$. Putting $r_1 = n - 20k$ (from the top equation) into the bottom one gives
$k = 11(n - 20k) + r_2 = 11n - 220k + r_2$ and solving for $r_2$: $r_2 = 221k - 11n$. If we put the values back in, this says $1 = 221 \cdot 100$ in the ring $\mathbb{Z}_{2009}$. Thus $(100)^{-1} = 221$.

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Second Exam C (solutions) March 27, 2024

1. For each of the following first-order recurrence relations, find the solution that satisfies the given initial condition.

   (a) $a_n = 8a_{n-1}\,,\ \ n \geq 1\,,$
   $a_0 = 3\,.$

**Ans:** Geometric progression: $a_n = (3)8^n$

   (b) $a_n = a_{n-1} + 9\,,\ \ n \geq 1\,,$
   $a_0 = 4\,.$

**Ans:** Arithmetic progression: $a_n = 4 + 9n$

   (c) $a_n = (n^6 + 1)a_{n-1}\,,\ \ n \geq 1\,,$
   $a_0 = 5\,.$

**Ans:** Successive multiplications: $a_n = 5(1^6 + 1)(2^6 + 1)\cdots(n^6 + 1)$.

   (d) $a_n = a_{n-1} + n^2 9^n\,,\ \ n \geq 1\,,$
   $a_0 = 6\,.$

**Ans:** Successive additions: $a_n = 6 + (1^2)9^1 + (2^2)9^2 + \cdots + n^2 9^n$

2. For the following second-order recurrence relation, find (i) the characteristic equation and its roots, (ii) the general solution, and (iii) the solution that satisfies the initial conditions.

$$a_n - 8a_{n-1} + 16a_{n-2} = 0\,,\ \ n \geq 2\,,$$
$$a_0 = 1 \ \text{ and } \ a_1 = 9,.$$

**Ans:** (i) $r^2 - 8r + 16 = 0$ has roots 4 and 4 (repeated root).

(ii) General solution: $C_1 4^n + C_2 n 4^n$.

(iii) The ICs, $C_1 = 1$ and $4C_1 + 4C_2 = 9$, give $C_1 = 1$ and $C_2 = 5/4$. Thus:

$$a_n = 4^n + \frac{5}{4} n 4^n$$

3. For the following second-order recurrence relation, find (i) the characteristic equation and its roots, (ii) the general solution, and (iii) the solution that satisfies the initial conditions.

$$a_n - 2a_{n-1} + 17a_{n-2} = 0\,, \quad n \geq 2\,,$$
$$a_0 = 0 \quad \text{and} \quad a_1 = 6.$$

**Ans:** (i) $r^2 - 2r + 17 = 0$ has roots $r = 1 \pm 4i$.

(ii) General solution: $C_1(1+4i)^n + C_2(1-4i)^n$.

(iii) The ICs, $C_1 + C_2 = 0$ and $C_1(1+4i) + C_2(1-4i) = 6$, give $C_1 = \dfrac{3}{4i}$ and $C_2 = \dfrac{-3}{4i}$. Thus: $a_n = \dfrac{3}{4i}(1+4i)^n - \dfrac{3}{4i}(1-4i)^n$.

4. For the following nonhomogeneous recurrence relation with initial conditions find (i) the homogeneous solution $a_n^{(h)}$, (ii) a particular solution $a_n^{(p)}$, and (iii) the solution that satisfies the initial conditions.

$$a_n - 7a_{n-1} + 10a_{n-2} = 4\,, \quad n \geq 2\,,$$
$$a_0 = 0 \quad \text{and} \quad a_1 = 0\,.$$

**Ans:** (i) The homogeneous solution is $a_n^{(h)} = C_1 2^n + C_2 5^n$.

(ii) A particular solution exists in the form $a_n^{(p)} = A$ for some constant $A$. The recurrence relation gives $A - 7A + 10A = 4$, so $a_n^{(p)} = A = 1$.

(iii) The general solution is therefore $C_1 2^n + C_2 5^n + 1$. The ICs, $C_1 + C_2 + 1 = 0$ and $2C_1 + 5C_2 + 1 = 0$, give $C_1 = -4/3$ and $C_2 = 1/3$. Thus:

$$a_n = (-4/3)2^n + (1/3)5^n + 1$$

5. For the following recurrence relation problem, find the generating function $F(x)$ for the sequence $a_n$ without actually finding $a_n$.

$a_n - 2a_{n-1} - 4a_{n-2} = 5\,,\ \ n \geq 2\,.$
$a_0 = 3,\ \ a_1 = 6.$

**Ans:** Multiplying by $x^n$ and summing gives

$$\sum_{n=2}^{\infty} a_n x^n - 2\sum_{n=2}^{\infty} a_{n-1}x^n - 4\sum_{n=2}^{\infty} a_{n-2}x^n = 5\sum_{n=2}^{\infty} x^n$$

This gives us

$$F(x) - 3 - 6x - 2x(F(x) - 3) - 4x^2 F(x) = \frac{5x^2}{1-x}.$$

or

$$(1 - 2x - 4x^2)F(x) = 3 + \frac{5x^2}{1-x} \quad \text{whence} \quad F(x) = \frac{3 + \dfrac{5x^2}{1-x}}{1 - 2x - 4x^2}$$

6. Do the following for the given rings.

(a) The factorization of 2728 into primes is $2728 = 2^3 \cdot 11 \cdot 31$. Find (i) the number of units and (ii) the number of proper zero divisors in the ring $\mathbb{Z}_{2728}$. Your final answers must be **completely** simplified.

**Ans:** Units: $\phi(2728) = 2^3 \cdot 11 \cdot 31 \left(\frac{1}{2}\right)\left(\frac{10}{11}\right)\left(\frac{30}{31}\right) = 1200$

Proper zero divisors: $2728 - 1200 - 1 = 1527$

(b) In the ring $\mathbb{Z}_{2011}$ find $(100)^{-1}$. Your final answer must be an explicit element of $\mathbb{Z}_{2011}$. (Note: this is **not** the same ring as the one in part (a).)

**Ans:** The Euclidean algorithm gives:

$$\begin{cases} 2011 = 20(100) + 11 \\ \ \ 100 = \ \ 9(11) + \ 1 \end{cases} \quad \text{or} \quad \begin{cases} n = 20k \ + r_1 \\ k = \ \ 9r_1 + r_2 \end{cases}$$

where $n = 2011$, $k = 100$, $r_1 = 11$, and $r_2 = 1$. Putting $r_1 = n - 20k$ (from the top equation) into the bottom one gives
$k = 9(n - 20k) + r_2 = 9n - 180k + r_2$ and solving for $r_2$: $r_2 = 181k - 9n$. If we put the values back in, this says $1 = 181 \cdot 100$ in the ring $\mathbb{Z}_{2011}$. Thus $(100)^{-1} = 181$.