Math 3103 Combinatorics (Luecking) NAME:____________________________

(Please print clearly)

First Exam A (solutions) February 19, 2024

Note: If I ask for an answer "in elementary form", that means you must write it using only numbers and the operations of addition, subtraction, multiplication, division, powers and factorials. When I **do not** explicitly request this, you may also use $C(n,k)$, $\binom{n}{k}$, $P(n,k)$ and d_n (with explicit numbers, of course).

1. A contest has 5 prizes and 20 contestants. If all 5 prizes must be awarded to some contestant, how many possible ways could this be done under each of the following conditions. All these answers must be in elementary form.

   (a) The 5 prizes are all different, and no contestant may receive more than one.

Ans: Select 5 contestants and label each with a different prize: $P(20,5) = \dfrac{20!}{(20-5)!}$.

   (b) The 5 prizes are identical, and no contestant may receive more than one.

Ans: Select 5 elements from a set of 20: $C(20,5) = \dfrac{20!}{5!\,(20-5)!}$.

   (c) The 5 prizes are identical, and any contestant may receive any number of them.

Ans: Select 5 times, with repetition, from a set of size 20: $C(5+20-1,5) = \dfrac{24!}{5!\,(24-5)!}$

2. Answer the following questions about the 10-letter string `"HELLENISTS"`, which has some repeated letters: 2 `"E"`s, 2 `"L"`s and 2 `"S"`s. The remaining letters occur only once each. Both answers must be in elementary form.

   (a) How many different arrangements of this string are there?

Ans: $\dfrac{10!}{2!\,2!\,2!}$

   (b) How many of the arrangements in part (a) contain **none** of the substrings `"EE"`, `"LL"` and `"SS"`?

Ans: c_1 = 'contains `"EE"`'; c_2 = 'contains `"LL"`'; c_3 = 'contains `"SS"`'.

$$S_1 = \binom{3}{1} N(c_1) = 3\left(\frac{9!}{2!\,2!}\right), \quad S_2 = \binom{3}{2} N(c_1c_2) = 3\left(\frac{8!}{2!}\right), \quad S_3 = N(c_1c_2c_3) = 7!\,,$$

so

$$N(\bar{c}_1\bar{c}_2\bar{c}_3) = N - S_1 + S_2 - S_3 = \frac{10!}{2!\,2!\,2!} - 3\left(\frac{9!}{2!\,2!}\right) + 3\left(\frac{8!}{2!}\right) - 7!$$

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3. Answer the following questions about the 10-letter string "`U N T A I L O R E D`", which has no repeated letters. Both answers must be in elementary form.

   (a) How many arrangements of this string contain **at least one** of the substrings "`L A`", "`T I`", "`D O`" and "`R E`"?

Ans: c_1 = 'contains "`L A`"'; c_2 = 'contains "`T I`"'; c_3 = 'contains "`D O`"'; c_4 = 'contains "`R E`".

$S_1 = \binom{4}{1} N(c_1) = 4(9!\,)$, $S_2 = \binom{4}{2} N(c_1c_2) = 6(8!\,)$, $S_3 = \binom{4}{3} N(c_1c_2c_3) = 4(7!\,)$ and $S_4 = N(c_1c_2c_3c_4) = 6!\,$, so:

$$L_1 = S_1 - S_2 + S_3 - S_4 = 4(9!\,) - 6(8!\,) + 4(7!\,) - 6!\,.$$

   (b) How many arrangements of this string contain **exactly one** of the four substrings "`L A`", "`T I`", "`D O`" or "`R E`"?

Ans: $E_1 = S_1 - 2S_2 + 3S_3 - 4S_4 = 4(9!\,) - 2 \cdot 6(8!\,) + 3 \cdot 4(7!\,) - 4(6!\,)$.

4. Suppose 8 students drop their phones in a box before a test, and then the phones are returned randomly. Answer the following questions. Only part (a) is required to be in elementary form.

   (a) In how many ways will none of the students get their own phone? Write this answer in elementary form.

Ans: Number of derangements of 8 things: $d_8 = 8!\left(\frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} - \frac{1}{5!} + \frac{1}{6!} - \frac{1}{7!} + \frac{1}{8}\right)$

   (b) In how many ways will at least one of the students get their own phone?

Ans: Permutations that are not derangements: $8! - d_8$

   (c) In how many ways will exactly 3 of the students get their own phone?

Ans: Choose which 3 get their own, then derange the rest: $\binom{8}{3} d_5$

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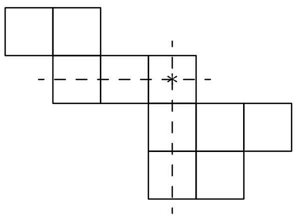
5. (a) Find the rook polynomial for the following chessboard C. Use the method that involves removing a square and then the product formula. Write the answer as a sum of numbers times different powers of x, similar to the one in part (b).



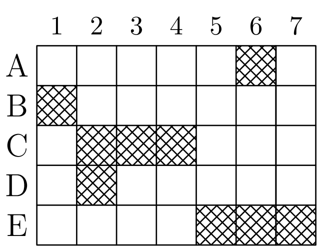
Ans: The first part of $r(C, x)$ below comes from removing the starred square, the second part from removing also the squares in the same row and column.

$$\begin{aligned} r(C, x) &= (1 + 4x + 3x^2)(1 + 5x + 4x^2) + x(1 + 3x + x^2)(1 + 2x) \\ &= (1 + 9x + 27x^2 + 31x^3 + 12x^2) + (x + 5x^2 + 7x^3 + 2x^4) \\ &= 1 + 10x + 32x^2 + 38x^3 + 14x^4 \,. \end{aligned}$$

(b) This table represents possible seating of 5 people in 7 seats. A shaded square represents a forbidden seat assignment. The rook polynomial of the shaded squares is

$$1 + 9x + 28x^2 + 36x^3 + 20x^4 + 4x^5$$

Using this information, how many allowable ways are there to seat all five people? (Your final answer should must be in elementary form.)



Ans: $P(7, 5) - 9P(6, 4) + 28P(5, 3) - 36P(4, 2) + 20P(3, 1) - 4P(2, 0)$

$$= \frac{7!}{2!} - 9\left(\frac{6!}{2!}\right) + 28\left(\frac{5!}{2!}\right) - 36\left(\frac{4!}{2!}\right) + 20\left(\frac{3!}{2!}\right) - 4\left(\frac{2!}{2!}\right)$$

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6. Consider the following equation. The variables w_1, w_2, w_3 and w_4 are integers that must satisfy the given conditions.

$$w_1 + w_2 + w_3 + w_4 = n$$

$$\begin{array}{ll} 20 \le w_1 & \longrightarrow x^{20} + x^{21} + \cdots = \dfrac{x^{20}}{1-x} \\ 0 \le w_2 \le 19 & \longrightarrow 1 + x + x^2 + \cdots + x^{19} = \dfrac{1-x^{20}}{1-x} \\ 10 \le w_3 \le 29 & \longrightarrow x^{10} + x^{11} + \cdots + x^{29} = \dfrac{x^{10}-x^{30}}{1-x} \\ 0 \le w_4 & \longrightarrow 1 + x + x^2 + \cdots = \dfrac{1}{1-x} \end{array}$$

(a) Write out the generating function for the number of solutions of this equation. Write your answer as a quotient in which the denominator is a power of $(1-x)$ and the numerator is a polynomial written out as a sum of powers of x.

Ans: The part of the generating function coming from each variable is given above. The generating function for the given problem is the product of these:

$$F(x) = \frac{x^{30} - 2x^{50} + x^{70}}{(1-x)^4}.$$

(b) Use the result in part (a) to find the number of solutions when $n = 85$.

Ans: $F(x) = \left(x^{30} - 2x^{50} + x^{70}\right) \sum_{j=0}^{\infty} \binom{j+3}{j} x^j$, and the terms that produce x^{85} are:

$$x^{30}\binom{58}{55}x^{55} - 2x^{50}\binom{38}{35}x^{35} + x^{70}\binom{18}{15}x^{15} = \left[\binom{58}{55} - 2\binom{38}{35} + \binom{18}{15}\right]x^{75},$$

so the answer is: $\binom{58}{55} - 2\binom{38}{35} + \binom{18}{15}$

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Math 3103 Combinatorics (Luecking) NAME:________________
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First Exam B (solutions) February 19, 2024

Note: If I ask for an answer "in elementary form", that means you must write it using only numbers and the operations of addition, subtraction, multiplication, division, powers and factorials. When I **do not** explicitly request this, you may also use $C(n,k)$, $\binom{n}{k}$, $P(n,k)$ and d_n (with explicit numbers, of course).

1. A contest has 6 prizes and 21 contestants. If all 6 prizes must be awarded to some contestant, how many possible ways could this be done under each of the following conditions. All these answers must be in elementary form.

(a) The 6 prizes are all different, and no contestant may receive more than one.

Ans: Select 6 contestants and label each with a different prize: $P(21,6) = \dfrac{21!}{(21-6)!}$.

(b) The 6 prizes are identical, and no contestant may receive more than one.

Ans: Select 6 elements from a set of 21: $C(21,6) = \dfrac{21!}{6!\ (21-6)!}$.

(c) The 6 prizes are identical, and any contestant may receive any number of them.

Ans: Select 6 times, with repetition, from a set of size 21: $C(6+21-1,6) = \dfrac{26!}{6!\ (26-6)!}$

2. Answer the following questions about the 11-letter string "HELLENISTIC", which has some repeated letters: 2 "E"s, 2 "L"s and 2 "I"s. The remaining letters occur only once each. Both answers must be in elementary form.

(a) How many different arrangements of this string are there?

Ans: $\dfrac{11!}{2!\ 2!\ 2!}$

(b) How many of the arrangements in part (a) contain **none** of the substrings "EE", "LL" and "II"?

Ans: c_1 = 'contains "EE"'; c_2 = 'contains "LL"'; c_3 = 'contains "II"'.

$$S_1 = \binom{3}{1} N(c_1) = 3\left(\frac{10!}{2!\ 2!}\right), \quad S_2 = \binom{3}{2} N(c_1c_2) = 3\left(\frac{9!}{2!}\right), \quad S_3 = N(c_1c_2c_3) = 8!\,,$$

so

$$N(\bar{c}_1\bar{c}_2\bar{c}_3) = N - S_1 + S_2 - S_3 = \frac{11!}{2!\ 2!\ 2!} - 3\left(\frac{10!}{2!\ 2!}\right) + 3\left(\frac{9!}{2!}\right) - 8!$$

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3. Answer the following questions about the 11-letter string "`FLUORINATED`", which has no repeated letters. Both answers must be in elementary form.

   (a) How many arrangements of this string contain **at least one** of the substrings "`LA`", "`TI`", "`DO`" and "`RE`"?

Ans: c_1 = 'contains "`LA`"'; c_2 = 'contains "`TI`"'; c_3 = 'contains "`DO`"'; c_4 = 'contains "`RE`".

$S_1 = \binom{4}{1} N(c_1) = 4(10!\,),\ S_2 = \binom{4}{2} N(c_1 c_2) = 6(9!\,),\ S_3 = \binom{4}{3} N(c_1 c_2 c_3) = 4(8!\,)$ and $S_4 = N(c_1 c_2 c_3 c_4) = 7!$, so:

$$L_1 = S_1 - S_2 + S_3 - S_4 = 4(10!\,) - 6(9!\,) + 4(8!\,) - 7!\,.$$

   (b) How many arrangements of this string contain **exactly one** of the four substrings "`LA`", "`TI`", "`DO`" or "`RE`"?

Ans: $E_1 = S_1 - 2S_2 + 3S_3 - 4S_4 = 4(10!\,) - 2 \cdot 6(9!\,) + 3 \cdot 4(8!\,) - 4(7!\,)$.

4. Suppose 8 students turn in term papers without putting their names on them, and then the papers are returned randomly. Answer the following questions. Only part (a) is required to be in elementary form.

   (a) In how many ways will none of the students get their own paper? Write this answer in elementary form.

Ans: Number of derangements of 8 things: $d_8 = 8!\left(\frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} - \frac{1}{5!} + \frac{1}{6!} - \frac{1}{7!} + \frac{1}{8}\right)$

   (b) In how many ways will at least one of the students get their own paper?

Ans: Permutations that are not derangements: $8! - d_8$

   (c) In how many ways will exactly 3 of the students get their own paper?

Ans: Choose which 3 get their own, then derange the rest: $\binom{8}{3} d_5$

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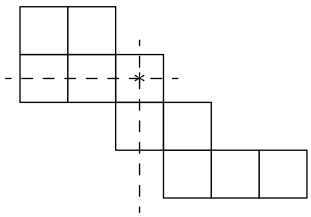
5. (a) Find the rook polynomial for the following chessboard C. Use the method that involves removing a square and then the product formula. Write the answer as a sum of numbers times different powers of x, similar to the one in part (b).



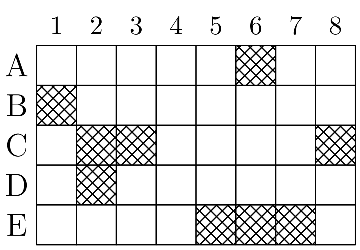
Ans: The first part of $r(C, x)$ below comes from removing the starred square, the second part from removing also the squares in the same row and column.

$$
\begin{aligned}
r(C, x) &= (1 + 4x + 2x^2)(1 + 5x + 5x^2) + x(1 + 2x)(1 + 4x + 2x^2) \\
&= (1 + 9x + 27x^2 + 30x^3 + 10x^4) + (x + 6x^2 + 20x^3 + 4x^4) \\
&= 1 + 10x + 33x^2 + 40x^3 + 14x^4 .
\end{aligned}
$$

(b) This table represents possible seating of 5 people in 8 seats. A shaded square represents a forbidden seat assignment. The rook polynomial of the shaded squares is

$$1 + 9x + 28x^2 + 36x^3 + 20x^4 + 4x^5$$

Using this information, how many allowable ways are there to seat all five people? (Your final answer should must be in elementary form.)



Ans: $P(8, 5) - 9P(7, 4) + 28P(6, 3) - 36P(5, 2) + 20P(4, 1) - 4P(3, 0)$

$$= \frac{8!}{3!} - 9\left(\frac{7!}{3!}\right) + 28\left(\frac{6!}{3!}\right) - 36\left(\frac{5!}{3!}\right) + 20\left(\frac{4!}{3!}\right) - 4\left(\frac{3!}{3!}\right)$$

Don't write in this box

6. Consider the following equation. The variables w_1, w_2, w_3 and w_4 are integers that must satisfy the given conditions.

$$w_1 + w_2 + w_3 + w_4 = n$$

$$10 \le w_1 \longrightarrow x^{10} + x^{11} + \cdots = \frac{x^{10}}{1-x}$$

$$0 \le w_2 \le 19 \longrightarrow 1 + x + x^2 + \cdots + x^{19} = \frac{1-x^{20}}{1-x}$$

$$10 \le w_3 \le 29 \longrightarrow x^{10} + x^{11} + \cdots + x^{29} = \frac{x^{10}-x^{30}}{1-x}$$

$$0 \le w_4 \longrightarrow 1 + x + x^2 + \cdots = \frac{1}{1-x}$$

(a) Write out the generating function for the number of solutions of this equation. Write your answer as a quotient in which the denominator is a power of $(1-x)$ and the numerator is a polynomial written out as a sum of powers of x.

Ans: The part of the generating function coming from each variable is given above. The generating function for the given problem is the product of these:

$$F(x) = \frac{x^{20} - 2x^{40} + x^{60}}{(1-x)^4}.$$

(b) Use the result in part (a) to find the number of solutions when $n = 85$.

Ans: $F(x) = \left(x^{20} - 2x^{40} + x^{60}\right) \sum_{j=0}^{\infty} \binom{j+3}{j} x^j$, and the terms that produce x^{85} are:

$$x^{20}\binom{68}{65}x^{65} - 2x^{40}\binom{48}{45}x^{45} + x^{60}\binom{28}{25}x^{25} = \left[\binom{68}{65} - 2\binom{48}{45} + \binom{28}{25}\right]x^{85},$$

so the answer is: $\binom{68}{65} - 2\binom{48}{45} + \binom{28}{25}$

Don't write in this box

Math 3103 Combinatorics (Luecking) NAME:______________________
(Please print clearly)

First Exam C (solutions) February 19, 2024

Note: If I ask for an answer "in elementary form", that means you must write it using only numbers and the operations of addition, subtraction, multiplication, division, powers and factorials. When I **do not** explicitly request this, you may also use $C(n, k)$, $\binom{n}{k}$, $P(n, k)$ and d_n (with explicit numbers, of course).

1. A contest has 7 prizes and 22 contestants. If all 7 prizes must be awarded to some contestant, how many possible ways could this be done under each of the following conditions. All these answers must be in elementary form.

   (a) The 7 prizes are all different, and no contestant may receive more than one.

Ans: Select 7 contestants and label each with a different prize: $P(22, 7) = \dfrac{22!}{(22-7)!}$.

   (b) The 7 prizes are identical, and no contestant may receive more than one.

Ans: Select 7 elements from a set of 22: $C(22, 7) = \dfrac{22!}{7!\,(22-7)!}$.

   (c) The 7 prizes are identical, and any contestant may receive any number of them.

Ans: Select 7 times, with repetition, from a set of size 22: $C(7 + 22 - 1, 7) = \dfrac{28!}{7!\,(28-7)!}$

2. Answer the following questions about the 12-letter string "HELIOSPHERIC", which has some repeated letters: 2 "E"s, 2 "H"s and 2 "I"s. The remaining letters occur only once each. Both answers must be in elementary form.

   (a) How many different arrangements of this string are there?

Ans: $\dfrac{12!}{2!\,2!\,2!}$

   (b) How many of the arrangements in part (a) contain **none** of the substrings "EE", "HH" and "II"?

Ans: c_1 = 'contains "EE"'; c_2 = 'contains "HH"'; c_3 = 'contains "II"'.

$$S_1 = \binom{3}{1} N(c_1) = 3\left(\frac{11!}{2!\,2!}\right), \quad S_2 = \binom{3}{2} N(c_1c_2) = 3\left(\frac{10!}{2!}\right), \quad S_3 = N(c_1c_2c_3) = 9!\,,$$

so

$$N(\bar{c}_1\bar{c}_2\bar{c}_3) = N - S_1 + S_2 - S_3 = \frac{12!}{2!\,2!\,2!} - 3\left(\frac{11!}{2!\,2!}\right) + 3\left(\frac{10!}{2!}\right) - 9!$$

Don't write in this box

3. Answer the following questions about the 8-letter string "`TAILORED`", which has no repeated letters. Both answers must be in elementary form.

   (a) How many arrangements of this string contain **at least one** of the substrings "`LA`", "`TI`", "`DO`" and "`RE`"?

Ans: c_1 = 'contains "`LA`"'; c_2 = 'contains "`TI`"'; c_3 = 'contains "`DO`"'; c_4 = 'contains "`RE`".

$S_1 = \binom{4}{1} N(c_1) = 4(7!\,)$, $S_2 = \binom{4}{2} N(c_1c_2) = 6(6!\,)$, $S_3 = \binom{4}{3} N(c_1c_2c_3) = 4(5!\,)$ and $S_4 = N(c_1c_2c_3c_4) = 4!$, so:

$$L_1 = S_1 - S_2 + S_3 - S_4 = 4(7!\,) - 6(6!\,) + 4(5!\,) - 4!\,.$$

   (b) How many arrangements of this string contain **exactly one** of the four substrings "`LA`", "`TI`", "`DO`" or "`RE`"?

Ans: $E_1 = S_1 - 2S_2 + 3S_3 - 4S_4 = 4(7!\,) - 2 \cdot 6(6!\,) + 3 \cdot 4(5!\,) - 4(4!\,)$.

4. Suppose 7 students turn in term papers without putting their names on them, and then the papers are returned randomly. Answer the following questions. Only part (a) is required to be in elementary form.

   (a) In how many ways will none of the students get their own paper? Write this answer in elementary form.

Ans: Number of derangements of 7 things: $d_7 = 7!\left(\frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} - \frac{1}{5!} + \frac{1}{6!} - \frac{1}{7!}\right)$

   (b) In how many ways will at least one of the students get their own paper?

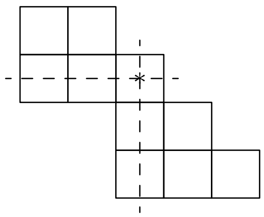
Ans: Permutations that are not derangements: $7! - d_7$

   (c) In how many ways will exactly 3 of the students get their own paper?

Ans: Choose which 3 get their own, then derange the rest: $\binom{7}{3} d_4$

Don't write in this box

5. (a) Find the rook polynomial for the following chessboard C. Use the method that involves removing a square and then the product formula. Write the answer as a sum of numbers times different powers of x, similar to the one in part (b).



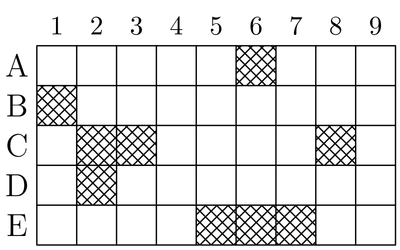
Ans: The first part of $r(C, x)$ below comes from removing the starred square, the second part from removing also the squares in the same row and column.

$$\begin{aligned} r(C, x) &= (1 + 4x + 2x^2)(1 + 5x + 4x^2) + x(1 + 2x)(1 + 3x + x^2) \\ &= (1 + 9x + 23x^2 + 26x^3 + 8x^4) + (x + 5x^2 + 7x^3 + 2x^4) \\ &= 1 + 10x + 28x^2 + 33x^3 + 12x^4 . \end{aligned}$$

(b) This table represents possible seating of 5 people in 9 seats. A shaded square represents a forbidden seat assignment. The rook polynomial of the shaded squares is

$$1 + 9x + 28x^2 + 36x^3 + 20x^4 + 4x^5$$

Using this information, how many allowable ways are there to seat all five people? (Your final answer should must be in elementary form.)



Ans: $P(9, 5) - 9P(8, 4) + 28P(7, 3) - 36P(6, 2) + 20P(5, 1) - 4P(4, 0)$

$$= \frac{9!}{4!} - 9\left(\frac{8!}{4!}\right) + 28\left(\frac{7!}{4!}\right) - 36\left(\frac{6!}{4!}\right) + 20\left(\frac{5!}{4!}\right) - 4\left(\frac{4!}{4!}\right)$$

Don't write in this box

6. Consider the following equation. The variables w_1, w_2, w_3 and w_4 are integers that must satisfy the given conditions.

$$w_1 + w_2 + w_3 + w_4 = n$$

$$\begin{array}{ll} 10 \le w_1 & \longrightarrow x^{10} + x^{11} + \cdots = \dfrac{x^{10}}{1-x} \\ 0 \le w_2 \le 19 & \longrightarrow 1 + x + x^2 + \cdots + x^{19} = \dfrac{1-x^{20}}{1-x} \\ 10 \le w_3 \le 29 & \longrightarrow x^{10} + x^{11} + \cdots + x^{29} = \dfrac{x^{10}-x^{30}}{1-x} \\ 0 \le w_4 & \longrightarrow 1 + x + x^2 + \cdots = \dfrac{1}{1-x} \end{array}$$

(a) Write out the generating function for the number of solutions of this equation. Write your answer as a quotient in which the denominator is a power of $(1-x)$ and the numerator is a polynomial written out as a sum of powers of x.

Ans: The part of the generating function coming from each variable is given above. The generating function for the given problem is the product of these:

$$F(x) = \frac{x^{20} - 2x^{40} + x^{60}}{(1-x)^4}.$$

(b) Use the result in part (a) to find the number of solutions when $n = 95$.

Ans: $F(x) = \left(x^{20} - 2x^{40} + x^{60}\right) \sum_{j=0}^{\infty} \binom{j+3}{j} x^j$, and the terms that produce x^{95} are:

$$x^{20}\binom{78}{75}x^{75} - 2x^{40}\binom{58}{55}x^{55} + x^{60}\binom{38}{35}x^{35} = \left[\binom{78}{75} - 2\binom{58}{55} + \binom{38}{35}\right]x^{95},$$

so the answer is: $\binom{78}{75} - 2\binom{58}{55} + \binom{38}{35}$

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