MATH 3083 Linear Algebra (Luecking) NAME:________________________
(Please print clearly)

Third Exam (solutions) April 24, 2024

Note: while numeric expressions are not required to be simplified, vector and matrix expressions must be. Thus $(1/4)\begin{pmatrix} 2 & 2 & 2 & 2 \end{pmatrix}$ is incomplete, but $\begin{pmatrix} 2/4 & 2/4 & 2/4 & 2/4 \end{pmatrix}$ is fine.

1. (a) Let $A = \begin{pmatrix} 1 & 2 \\ -1 & -1 \\ 0 & 2 \\ -1 & 0 \end{pmatrix}$. Find a basis for $\mathcal{R}(A)^\perp$. ($\mathcal{R}(A)$ is the column space of A).

Ans: Find $\mathcal{N}(A^T)$:

$$\left(\begin{array}{cccc|c} 1 & -1 & 0 & -1 & 0 \\ 2 & -1 & 2 & 0 & 0 \end{array}\right) \xrightarrow{\text{3 EROs}} \left(\begin{array}{cccc|c} 1 & 0 & 2 & 1 & 0 \\ 0 & 1 & 2 & 2 & 0 \end{array}\right)$$

then, from $x_1 = -2x_3 - x_4$ and $x_2 = -2x_3 - 2x_4$ we get

$$(x_3 = 1,\ x_4 = 0) : \begin{pmatrix} -2 \\ -2 \\ 1 \\ 0 \end{pmatrix} \quad \text{and} \quad (x_3 = 0,\ x_4 = 0) : \begin{pmatrix} -1 \\ -2 \\ 0 \\ 1 \end{pmatrix}$$

(b) Let $\mathbf{b} = \begin{pmatrix} 1 \\ 1 \\ 2 \\ -1 \end{pmatrix}$. Find the least squares solution $\hat{\mathbf{x}}$ of $A\mathbf{x} = \mathbf{b}$.

Ans: Solve $A^{\mathrm{T}}A\mathbf{x} = A^{\mathrm{T}}\mathbf{b}$. That is

$$\begin{pmatrix} 3 & 3 \\ 3 & 9 \end{pmatrix} \mathbf{x} = \begin{pmatrix} 1 \\ 5 \end{pmatrix}$$

This gives $\hat{\mathbf{x}} = \begin{pmatrix} -1/3 \\ 2/3 \end{pmatrix}$.

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2. For the vectors $\mathbf{x} = \begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix}$ and $\mathbf{y} = \begin{pmatrix} 3 \\ -1 \\ 1 \end{pmatrix}$ in $\mathbb{R}^3$, do the following:

(a) Find the scalar product of $\mathbf{x}$ and $\mathbf{y}$. **Ans:** $\mathbf{x}^{\mathrm{T}}\mathbf{y} = 6 - 2 + 1 = 5$

(b) Find $\|\mathbf{x}\|$ and $\|\mathbf{y}\|$.

Ans: $\|x\| = \sqrt{4+4+1} = \sqrt{9} = 3$. $\|y\| = \sqrt{9+1+1} = \sqrt{11}$

(c) Find $\cos\theta$, where θ is the angle between $\mathbf{x}$ and $\mathbf{y}$. Do not find θ itself.

Ans: $\cos\theta = \dfrac{\mathbf{x}^{\mathrm{T}}\mathbf{y}}{\|\mathbf{x}\|\,\|\mathbf{y}\|} = \dfrac{5}{3\sqrt{11}}$.

(d) Find a **unit vector** in $\mathbb{R}^3$ that is orthogonal to both $\mathbf{x}$ and $\mathbf{y}$.

Ans: Solve the system: $\mathbf{x}^{\mathrm{T}}\mathbf{z} = 0$ and $\mathbf{y}^{\mathrm{T}}\mathbf{z} = 0$ for $\mathbf{z}$:

$$\left(\begin{array}{ccc|c} 2 & 2 & 1 & 0 \\ 3 & -1 & 1 & 0 \end{array}\right) \xrightarrow{\text{4 EROs}} \left(\begin{array}{ccc|c} 1 & 0 & 3/8 & 0 \\ 0 & 1 & 1/8 & 0 \end{array}\right)$$

If we set $z_3 = 8$ we get $(-3 \quad -1 \quad 8)^T$ Divide this by its norm to get the unit vector

$$\begin{pmatrix} -3/\sqrt{74} \\ -1/\sqrt{74} \\ 8/\sqrt{74} \end{pmatrix}$$

3. Using the scalar product on $\mathbb{R}^4$ and the two vectors $\mathbf{x} = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$ and $\mathbf{y} = \begin{pmatrix} -1 \\ 1 \\ 1 \\ -1 \end{pmatrix}$

answer the following.

(a) Is the set $\{\mathbf{x}, \mathbf{y}\}$ orthogonal? Yes ($\mathbf{x} \perp \mathbf{y}$)

Is the set $\{\mathbf{x}, \mathbf{y}\}$ orthonormal? No ($\|\mathbf{x}\| \neq 1$)

(b) Find the projection of $\mathbf{z} = \begin{pmatrix} 0 \\ 1 \\ 2 \\ 1 \end{pmatrix}$ onto $\operatorname{Span}(\mathbf{x}, \mathbf{y})$.

Ans: We can add the projections of $\mathbf{z}$ onto each of $\mathbf{x}$ and $\mathbf{y}$ because they are orthogonal:

$$\frac{\mathbf{z}^{\mathrm{T}}\mathbf{x}}{\mathbf{x}^{\mathrm{T}}\mathbf{x}}\mathbf{x} + \frac{\mathbf{z}^{\mathrm{T}}\mathbf{y}}{\mathbf{y}^{\mathrm{T}}\mathbf{y}}\mathbf{y} = \frac{4}{4}\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} + \frac{2}{4}\begin{pmatrix} -1 \\ 1 \\ 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 1/2 \\ 3/2 \\ 3/2 \\ 1/2 \end{pmatrix}$$

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4. Apply the Gram-Schmidt process to $\mathbf{x}_1 = \begin{pmatrix} 1 \\ 1 \\ 0 \\ 1 \end{pmatrix}$, $\mathbf{x}_2 = \begin{pmatrix} 1 \\ -1 \\ 0 \\ 1 \end{pmatrix}$, $\mathbf{x}_3 = \begin{pmatrix} 1 \\ 0 \\ 1 \\ 2 \end{pmatrix}$ (in that order) to obtain an orthonormal set $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3$ with the same span as $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3$.

Ans: $\mathbf{v}_1 = \mathbf{x}_1 = \begin{pmatrix} 1 \\ 1 \\ 0 \\ 1 \end{pmatrix}$, $\quad \mathbf{x}_2 - \dfrac{\mathbf{x}_2^{\mathrm{T}}\mathbf{v}_1}{\mathbf{v}_1^{\mathrm{T}}\mathbf{v}_1}\mathbf{v}_1 = \begin{pmatrix} 1 \\ -1 \\ 0 \\ 1 \end{pmatrix} - \dfrac{1}{3}\begin{pmatrix} 1 \\ 1 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 2/3 \\ -4/3 \\ 0 \\ 2/3 \end{pmatrix}$.

Multiply by 3/2 and set $\mathbf{v}_2 = \begin{pmatrix} 1 \\ -2 \\ 0 \\ 1 \end{pmatrix}$.

$$\mathbf{x}_3 - \frac{\mathbf{x}_3^{\mathrm{T}}\mathbf{v}_1}{\mathbf{v}_1^{\mathrm{T}}\mathbf{v}_1}\mathbf{v}_1 - \frac{\mathbf{x}_3^{\mathrm{T}}\mathbf{v}_2}{\mathbf{v}_2^{\mathrm{T}}\mathbf{v}_2}\mathbf{v}_2 = \begin{pmatrix} 1 \\ 0 \\ 1 \\ 2 \end{pmatrix} - \frac{3}{3}\begin{pmatrix} 1 \\ 1 \\ 0 \\ 1 \end{pmatrix} - \frac{3}{6}\begin{pmatrix} 1 \\ -2 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} -1/2 \\ 0 \\ 1 \\ 1/2 \end{pmatrix}.$$

Multiply by 2 and set $\mathbf{v}_3 = \begin{pmatrix} -1 \\ 0 \\ 2 \\ 1 \end{pmatrix}$. Then obtain orthonormal vectors by $\mathbf{u}_j = \dfrac{1}{\|\mathbf{v}_j\|}\mathbf{v}_j$. This produces

$$\mathbf{u}_1 = \begin{pmatrix} 1/\sqrt{3} \\ 1/\sqrt{3} \\ 0 \\ 1/\sqrt{3} \end{pmatrix}, \mathbf{u}_2 = \begin{pmatrix} 1/\sqrt{6} \\ -2/\sqrt{6} \\ 0 \\ 1/\sqrt{6} \end{pmatrix}, \mathbf{u}_3 = \begin{pmatrix} -1/\sqrt{6} \\ 0 \\ 2/\sqrt{6} \\ 1/\sqrt{6} \end{pmatrix}.$$

5. Find all eigenvalues of the following matrices. (Do *not* find any eigenvectors *nor* any eigenspaces.)

(a) $A = \begin{pmatrix} 1 & 4 \\ 1 & 1 \end{pmatrix}$. **Ans:** $|A - \lambda I| = (1-\lambda)^2 - 4 = \lambda^2 - 2\lambda - 3$ so $\lambda = -1, 3$

(b) $B = \begin{pmatrix} 3 & 1 & 1 \\ 0 & 2 & 4 \\ 0 & 1 & 2 \end{pmatrix}$.

Ans: $\begin{vmatrix} 3-\lambda & 1 & 1 \\ 0 & 2-\lambda & 4 \\ 0 & 1 & 2-\lambda \end{vmatrix} = (3-\lambda)\begin{vmatrix} 2-\lambda & 4 \\ 1 & 2-\lambda \end{vmatrix} = (3-\lambda)[(2-\lambda)^2 - 4]$

This simplifies to $(3-\lambda)(\lambda^2 - 4\lambda) = 0$, giving $\lambda = 3, 0, 4$.

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6. The matrix $A = \begin{pmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{pmatrix}$ satisfies $\det(A - \lambda I) = (4-\lambda)(\lambda-1)^2$.

(a) Find all the eigenvalues of A **and** the eigenspace of each eigenvalue.

(b) Answer the following questions: Is A diagonalizable? Why or why not?

Ans: (a) $(4-\lambda)(\lambda-1)^2 = 0$ gives eigenvalues $\lambda = 4, 1, 1$.

$\lambda = 4$: the matrix $A - 4I = \begin{pmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{pmatrix} \xrightarrow{\text{6 EROs}} \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix}$.

So, $x_1 = x_3$, $x_2 = x_3$ and the eigenspace consists of all $\begin{pmatrix} \alpha \\ \alpha \\ \alpha \end{pmatrix}$, $\alpha \in \mathbb{R}$.

$\lambda = 1$: the matrix $A - 1I = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} \xrightarrow{\text{2 EROs}} \begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$.

So, $x_1 = -x_2 - x_3$ and the eigenspace consists of all $\begin{pmatrix} -\alpha - \beta \\ \alpha \\ \beta \end{pmatrix}$, $\alpha, \beta \in \mathbb{R}$.

(b) A is diagonalizable because the dimensions of the eigenspaces add up to 3. Another valid reason: because A is symmetric. The third possible reason is to construct S out of eigenvectors, find S^{-1}, and then verify that $S^{-1}AS$ is diagonal. Obviously, we avoid doing that unless we actually need all that info.

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