MATH 3083 Linear Algebra (Luecking) NAME:________________________
(Please print clearly)

Second Exam (solutions) March 13, 2024

1. For the following sets, state whether it is a subspace of $\mathbb{R}^3$ (that is, answer 'yes' or 'no').
   - If you answer 'yes', either provide a matrix whose *nullspace* is that set, or provide a set of vectors whose *span* is that set.
   - If you answer 'no' give an example (using vectors made of actual numbers) that violates one of the closure requirements.

   (a) The set $\left\{ \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \middle| \, x_1 \le 0 \text{ and } x_3 \le 0 \right\}$.

Ans: No, this contains $\begin{pmatrix} -1 \\ 0 \\ 0 \end{pmatrix}$ but not $(-2) \begin{pmatrix} -1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix}$.

   (b) The set $\left\{ \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \middle| \, x_3 = -x_1 - 4x_2 \text{ and } x_2 = 3x_1 \right\}$.

Ans: Yes, this is the nullspace of $\begin{pmatrix} 1 & 4 & 1 \\ 3 & -1 & 0 \end{pmatrix}$.

   (c) The set $\left\{ \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \middle| \, x_2 = x_3 \text{ and } x_1 x_2 = 0 \right\}$.

Ans: No, this contains $\mathbf{x} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ and $\mathbf{y} = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$ but not $\mathbf{x} + \mathbf{y} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$.

   (d) The set $\left\{ \begin{pmatrix} \alpha \\ -2\alpha + 3\beta \\ 4\beta \end{pmatrix} \middle| \, \alpha, \beta \text{ in } \mathbb{R} \right\}$.

Ans: Yes, this is the span of $\left\{ \begin{pmatrix} 1 \\ -2 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 3 \\ 4 \end{pmatrix} \right\}$.

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2. For each of the following sets of vectors in $\mathbb{R}^3$, answer the following with 'yes' or 'no'.

(i) Does it span $\mathbb{R}^3$?

(ii) Is it independent?

(iii) Is it a basis for $\mathbb{R}^3$?

Justify your answers by reducing the *appropriate* matrix to *echelon form*.

(a) $\begin{pmatrix} 1 \\ 1 \\ -3 \end{pmatrix}, \begin{pmatrix} 2 \\ 3 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}$

Ans: The matrix with these columns: $\begin{pmatrix} 1 & 2 & 1 \\ 1 & 3 & 2 \\ -3 & -1 & 2 \end{pmatrix}$, reduces to $\begin{pmatrix} 1 & 2 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}$ in three steps. Then: (i) no, (ii) no and (iii) no.

(b) $\begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}, \begin{pmatrix} -2 \\ -1 \\ 1 \end{pmatrix}, \begin{pmatrix} 3 \\ 1 \\ -2 \end{pmatrix}$

Ans: The matrix with these columns: $\begin{pmatrix} 1 & -2 & 3 \\ 1 & -1 & 1 \\ -1 & 1 & -2 \end{pmatrix}$, reduces to $\begin{pmatrix} 1 & -2 & 3 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{pmatrix}$ in four steps. Then: (i) yes, (ii) yes and (iii) yes.

(c) $\begin{pmatrix} 0 \\ 1 \\ 3 \end{pmatrix}, \begin{pmatrix} 3 \\ -2 \\ -2 \end{pmatrix}$

Ans: The matrix with these columns: $\begin{pmatrix} 0 & 3 \\ 1 & -2 \\ 3 & -2 \end{pmatrix}$, reduces to $\begin{pmatrix} 1 & -2 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}$ in four steps. Then: (i) no, (ii) yes and (iii) no.

(d) $\begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 0 \\ -1 \\ 2 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ 3 \\ 2 \end{pmatrix}$

Ans: $\begin{pmatrix} 1 & 0 & 1 & 2 \\ 1 & -1 & 2 & 3 \\ 2 & 2 & 0 & 2 \end{pmatrix}$ reduces to $\begin{pmatrix} 1 & 0 & 1 & 2 \\ 0 & 1 & -1 & -1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$ in four steps. Then (i) no, (ii) no and (iii) no.

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3. Let $\mathcal{E} = \left[\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right]$ be the standard basis for $\mathbb{R}^3$, and let $\mathcal{B}$ be the ordered basis for $\mathbb{R}^3$ given by $\mathcal{B} = \left[\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix} \right]$. Find the change of basis matrix (transition matrix) from $\mathcal{E}$ to $\mathcal{B}$.

Ans: Invert the matrix with the columns from $\mathcal{B}$: row-reduce $\left(\begin{array}{ccc|ccc} 1 & 0 & 2 & 1 & 0 & 0 \\ 1 & 1 & 3 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right)$ until the left side is the identity (four steps) and read off the transition matrix from the right side: $\begin{pmatrix} 1 & 0 & -2 \\ -1 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix}$.

4. $A = \begin{pmatrix} 1 & -2 & 0 & -3 & 2 \\ -2 & 4 & 0 & 6 & -4 \\ -1 & 2 & 0 & 3 & -2 \\ 0 & 0 & 1 & 3 & 5 \end{pmatrix}$ has reduced echelon form: $\begin{pmatrix} 1 & -2 & 0 & -3 & 2 \\ 0 & 0 & 1 & 3 & 5 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$.

Do the following.

(a) Write down a basis for the column space of A:

Ans: The columns of A corresponding to leading 1s: $\left\{ \begin{pmatrix} 1 \\ -2 \\ -1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \right\}$

(b) Write down a basis for the row space of A:

Ans: The nonzero rows of the echelon form: $\{(1 \quad -2 \quad 0 \quad -3 \quad 2), (0 \quad 0 \quad 1 \quad 3 \quad 5))\}$

(c) Write down the rank and nullity of A:

Ans: The rank of A is 2, the size of the above bases. Its nullity is 3, the number of columns of the echelon form without leading ones (or $5 - 2$).

(d) Find a basis for the nullspace of A:

Ans: Solve $A\mathbf{x} = 0$ using the echelon form to get $x_1 = 2x_2 + 3x_4 - 2x_5$ and $x_3 = -3x_4 - 5x_5$. Set each free variable in turn to 1 (with the rest equal to 0) to get the basic solutions: $\left\{ \begin{pmatrix} 2 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 3 \\ 0 \\ -3 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -2 \\ 0 \\ -5 \\ 0 \\ 1 \end{pmatrix} \right\}$.

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5. Let T be the linear transformation from $\mathbb{R}^3$ to $\mathbb{R}^2$ given by
$T\left(\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}\right) = \begin{pmatrix} 2x_1 - 3x_2 + x_3 \\ 2x_1 - 3x_3 \end{pmatrix}$. Find the matrix A that satisfies $A\mathbf{x} = T(\mathbf{x})$ for all $\mathbf{x}$ in $\mathbb{R}^3$.

Ans: The columns of A are the three vectors
$T(\mathbf{e}_1) = \begin{pmatrix} 2 \\ 2 \end{pmatrix}$, $T(\mathbf{e}_2) = \begin{pmatrix} -3 \\ 0 \end{pmatrix}$, $T(\mathbf{e}_3) = \begin{pmatrix} 1 \\ -3 \end{pmatrix}$ so, $A = \begin{pmatrix} 2 & -3 & 1 \\ 2 & 0 & -3 \end{pmatrix}$.

6. Let $\mathcal{B} = \left[\begin{pmatrix} 2 \\ 3 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \end{pmatrix}\right]$, an ordered basis for $\mathbb{R}^2$. Let $\mathcal{C} = \left[\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}\right]$, an ordered basis for $\mathbb{R}^3$. Let L be the linear transformation from $\mathbb{R}^2$ to $\mathbb{R}^3$ defined by
$L\left(\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}\right) = \begin{pmatrix} x_1 \\ 2x_1 - x_2 \\ x_1 + x_2 \end{pmatrix}$.
Find the matrix that represents L relative to the bases $\mathcal{B}$ and $\mathcal{C}$.

Ans: Apply L to the vectors in $\mathcal{B}$: $L\begin{pmatrix} 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \\ 5 \end{pmatrix}$, $L\begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 3 \end{pmatrix}$, and find their coordinates relative to $\mathcal{C}$. I.e., solve the following two systems of equations

$$\begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \\ 5 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} \beta_1 \\ \beta_2 \\ \beta_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 3 \end{pmatrix}$$

Both these can be solved at the same time by row reducing:

$$\left(\begin{array}{ccc|cc} 1 & 0 & 1 & 2 & 1 \\ 1 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 5 & 3 \end{array}\right) \quad \text{to} \quad \left(\begin{array}{ccc|cc} 1 & 0 & 0 & -1 & -1 \\ 0 & 1 & 0 & 2 & 1 \\ 0 & 0 & 1 & 3 & 2 \end{array}\right)$$

The coordinate vectors are the last two columns, and thus the representing matrix is $A = \begin{pmatrix} -1 & -1 \\ 2 & 1 \\ 3 & 2 \end{pmatrix}$.

[Alternatively, multiply the matrix form of L by the two transition matrices ($\mathcal{B}$ to $\mathcal{E}$ on the right, $\mathcal{E}$ to $\mathcal{C}$ on the left).]

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