MATH 3083 Linear Algebra (Luecking) NAME:__________________________
(Please print clearly)

First Exam (solutions) February 14, 2024

1. For this system of equations:
$$\begin{aligned} x_1 + 2x_2 + x_3 - x_4 &= 2 \\ x_1 + 2x_2 + 2x_3 + 3x_4 &= 4 \\ 3x_1 + 6x_2 + 4x_3 + x_4 &= 8 \end{aligned}$$

   (a) Write down its **augmented** matrix.

   (b) Using *only* elementary row operations convert the augmented matrix to **reduced** echelon form. Please indicate what EROs you are performing.

   (c) State which variables are leading variables. State which variables are free variables.

   (d) Find all solutions of the system. Express your solutions (if any) as **column vectors** with parameters substituted for the free variables .

Ans:

(a) $\left(\begin{array}{cccc|c} 1 & 2 & 1 & -1 & 2 \\ 1 & 2 & 2 & 3 & 4 \\ 3 & 6 & 4 & 1 & 8 \end{array}\right)$ (b) $\xrightarrow[R_3 - 3R_1]{R_2 - R_1} \left(\begin{array}{cccc|c} 1 & 2 & 1 & -1 & 2 \\ 0 & 0 & 1 & 4 & 2 \\ 0 & 0 & 1 & 4 & 2 \end{array}\right) \xrightarrow[R_1 - R_2]{R_3 - R_2} \left(\begin{array}{cccc|c} 1 & 2 & 0 & -5 & 0 \\ 0 & 0 & 1 & 4 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$.

(c) x_1 and x_3 are lead variables; x_2 and x_4 are free.

(d) From the reduced form: $\left.\begin{aligned} x_1 + 2x_2 \qquad - 5x_4 &= 0 \\ x_3 + 4x_4 &= 2 \end{aligned}\right\} \longrightarrow \left\{\begin{aligned} x_1 &= -2x_2 + 5x_4 \\ x_3 &= 2 - 4x_4 \end{aligned}\right.$.

Set $x_2 = \alpha$ and $x_4 = \beta$, then solve for the others in terms of α and β:

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} -2\alpha + 5\beta \\ \alpha \\ 2 - 4\beta \\ \beta \end{pmatrix}, \text{ all } \alpha, \beta \text{ in } \mathbb{R}.$$

2. For each of the following augmented matrices, convert it to echelon form using *exactly one* elementary row operation, and then determine whether the corresponding system of equations has no solutions, one solution, or infinitely many solutions.

(a) $\left(\begin{array}{ccc|c} 1 & 2 & 3 & 1 \\ 0 & 1 & 4 & 3 \\ 2 & 4 & 7 & 3 \end{array}\right) \xrightarrow{R_3 - 2R_1} \left(\begin{array}{ccc|c} 1 & 2 & 3 & 1 \\ 0 & 1 & 4 & 3 \\ 0 & 0 & 1 & 1 \end{array}\right)$ (b) $\left(\begin{array}{ccc|c} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 3 \\ 1 & 3 & 4 & 4 \end{array}\right) \xrightarrow{R_1 \leftrightarrow R_3} \left(\begin{array}{ccc|c} 1 & 3 & 4 & 4 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{array}\right)$

(c) $\left(\begin{array}{ccc|c} 1 & 3 & 5 & 7 \\ 0 & 1 & 1 & 2 \\ 0 & 2 & 2 & 5 \end{array}\right) \xrightarrow{R_3 - 3R_2} \left(\begin{array}{ccc|c} 1 & 3 & 5 & 7 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 1 \end{array}\right)$ (d) $\left(\begin{array}{ccc|c} 1 & 0 & 4 & 4 \\ 0 & 2 & 5 & 4 \\ 0 & 0 & 1 & 0 \end{array}\right) \xrightarrow{(1/2)R_2} \left(\begin{array}{ccc|c} 1 & 0 & 4 & 4 \\ 0 & 1 & \frac{5}{2} & 2 \\ 0 & 0 & 1 & 0 \end{array}\right)$

Number of solutions: (a) one (b) infinitely many (c) none (d) one

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3. For the matrices $A = \begin{pmatrix} 1 & -1 & 1 \\ 2 & 1 & 2 \end{pmatrix}$, $B = \begin{pmatrix} 2 & 1 \\ 1 & 0 \\ 3 & 2 \end{pmatrix}$, $C = \begin{pmatrix} 1 & 3 \\ 1 & 2 \end{pmatrix}$ and $I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, find the following. When one is not possible, write 'not possible'.

(a) $CA + B^T$ (b) BA (c) $C - 2A$

(d) $C^T + C^2$ (e) AC (f) $B(C - I)$

Ans: (a) $\begin{pmatrix} 7 & 2 & 7 \\ 5 & 1 & 5 \end{pmatrix} + \begin{pmatrix} 2 & 1 & 3 \\ 1 & 0 & 2 \end{pmatrix} = \begin{pmatrix} 9 & 3 & 10 \\ 6 & 1 & 7 \end{pmatrix}$

(b) $\begin{pmatrix} 4 & -1 & 4 \\ 1 & -1 & 1 \\ 7 & -1 & 7 \end{pmatrix}$

(c) not possible

(d) $\begin{pmatrix} 1 & 1 \\ 3 & 2 \end{pmatrix} + \begin{pmatrix} 4 & 9 \\ 3 & 7 \end{pmatrix} = \begin{pmatrix} 5 & 10 \\ 6 & 9 \end{pmatrix}$

(e) not possible

(f) $\begin{pmatrix} 2 & 1 \\ 1 & 0 \\ 3 & 1 \end{pmatrix} \begin{pmatrix} 0 & 3 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 7 \\ 0 & 3 \\ 2 & 11 \end{pmatrix}$

4. For the matrix $A = \begin{pmatrix} 1 & 2 & 0 \\ 1 & 3 & -2 \\ 2 & 4 & 1 \end{pmatrix}$, find A^{-1} using elementary row operations.

Ans: $\left(\begin{array}{ccc|ccc} 1 & 2 & 0 & 1 & 0 & 0 \\ 1 & 3 & -2 & 0 & 1 & 0 \\ 2 & 4 & 1 & 0 & 0 & 1 \end{array}\right) \xrightarrow[R_3 - 2R_1]{R_2 - R_1} \left(\begin{array}{ccc|ccc} 1 & 2 & 0 & 1 & 0 & 0 \\ 0 & 1 & -2 & -1 & 1 & 0 \\ 0 & 0 & 1 & -2 & 0 & 1 \end{array}\right)$

$\xrightarrow{R_2 + 2R_3} \left(\begin{array}{ccc|ccc} 1 & 2 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & -5 & 1 & 2 \\ 0 & 0 & 1 & -1 & 0 & 1 \end{array}\right) \xrightarrow{R_1 - 2R_2} \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 11 & -2 & -4 \\ 0 & 1 & 0 & -5 & 1 & 2 \\ 0 & 0 & 1 & -2 & 0 & 1 \end{array}\right)$.

So $A^{-1} = \begin{pmatrix} 11 & -2 & -4 \\ -5 & 1 & 2 \\ -2 & 0 & 1 \end{pmatrix}$

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5. Given partitioned matrices $A = (A_1 \mid A_2)$ and $B = \left(\frac{B_1}{B_2}\right)$, satisfying

$$A_1B_1 = \begin{pmatrix} 4 & 0 \\ -3 & 2 \\ 4 & -1 \end{pmatrix} \text{ and } A_2B_2 = \begin{pmatrix} 1 & 1 \\ 2 & 3 \\ 4 & 6 \end{pmatrix},$$

answer the following:

(a) How many rows does A have?

Ans: 3, the same as the number of rows of A_1B_1 AND A_2B_2.

(b) How many columns does B have?

Ans: 2, the same as the number of columns of A_1B_1 and A_2B_2.

(c) Find AB.

Ans: $AB = A_1B_1 + A_2B_2 = \begin{pmatrix} 5 & 1 \\ -1 & 5 \\ 8 & 5 \end{pmatrix}$

6. Find the following determinants. Please produce completely simplified numerical results.

(a) $\begin{vmatrix} 9 & 5 & 7 & -8 \\ 2 & 2 & 0 & 1 \\ 2 & 2 & 0 & 1 \\ 11 & -16 & 10 & 9 \end{vmatrix}$ (b) $\begin{vmatrix} 4 & 7 & -12 & 2 \\ 0 & 2 & 8 & 0 \\ 5 & 2 & 9 & 3 \\ 0 & 0 & 2 & 0 \end{vmatrix}$ (c) $\begin{vmatrix} 2 & 3 & 0 & 0 \\ 1 & 3 & 0 & 0 \\ 1 & 1 & 3 & 1 \\ 3 & -1 & 4 & 2 \end{vmatrix}$

Ans: (a) Two identical rows, so the determinant is 0.

(b) Using cofactors of the last row, then cofactors of the second row, this determinant is

$$-2\begin{vmatrix} 4 & 7 & 2 \\ 0 & 2 & 0 \\ 5 & 2 & 3 \end{vmatrix} = -2(2)\begin{vmatrix} 4 & 2 \\ 5 & 3 \end{vmatrix} = -4(12-10) = -8\,.$$

(c) This determinant equals the product of the upper-left and lower-right determinants:

$$\begin{vmatrix} 2 & 3 \\ 1 & 3 \end{vmatrix} \cdot \begin{vmatrix} 3 & 1 \\ 4 & 2 \end{vmatrix} = (6-3)(6-4) = 6\,.$$

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