Groups and Burnside's Theorem

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MASC

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For this we also need to keep track of the length of each cycle. We keep track of this with another sort of generating function.

Let's work on the group for the equilateral triangle. For each element of the group let's substitute the variable x_1 for each 1-cycle (cycle of length 1), x_2 for each 2-cycle (cycle of length 2), x_3 for each 3-cycle, etc.:

$$\begin{array}{lcccccc} G: & (1)(2)(3) & (123) & (132) & (1)(23) & (2)(13) & (3)(12) \\ & x_1x_1x_1 & x_3 & x_3 & x_1x_2 & x_1x_2 & x_1x_2 \end{array}$$

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Then we add these all up, collect like terms and divide by the order of the group:

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On the next slide we do this for the rectangle's group.

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Here are the computations for the group of the square: $G = \{(1)(2)(3)(4), (1234),$ $(13)(24), (1432), (12)(34), (14)(23), (1)(24)(3), (2)(13)(4)\}$

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$$P_G(2, 2, 2, 2) = \frac{1}{4}(2^4 + 3 \cdot 2^2) = \frac{28}{4} = 7.$$

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$$\begin{aligned} P_G(r+w, r^2+w^2, r^3+w^3, r^4+w^4) &= \frac{1}{4}[(r+w)^4 + 3(r^2+w^2)^2] \\ &= r^4 + r^3w + 3r^2w^2 + rw^3 + w^4 \end{aligned}$$

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The number in front of each term tells us how many distinguishable colorings have that combination of colors: there is only one that uses 4 red, or 3 red and 1 white, or 3 white and 1 red, or 4 white.

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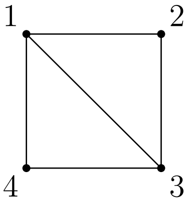

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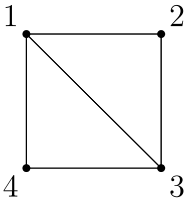

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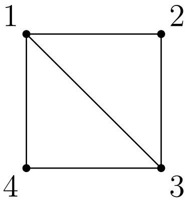

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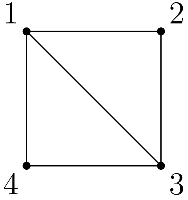

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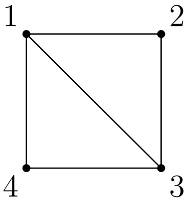

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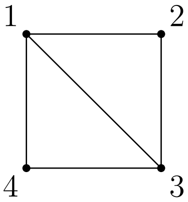

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For our current example:

$$\begin{aligned} P_G(r+w, r^2+w^2) &= \frac{(r+w)^4 + (r^2+w^2)^2 + 2(r+w)^2(r^2+w^2)}{4} \\ &= r^4 + 2r^3w + 3r^2w^2 + 2rw^3 + w^4, \end{aligned}$$

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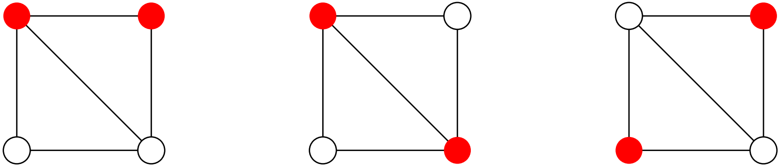

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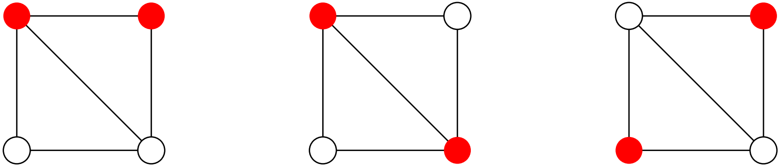

If there are three colors such as red, white and blue, we would substitute $x_1 = r + w + b$, $x_2 = r^2 + w^2 + b^2$, etc.

The term $3r^2w^2$ tells us there are 3 distinguishable colorings in which 2 vertices are red and 2 are white. And the term $2rw^3$ says there are 2 distinguishable colorings with 1 red and 3 white vertices. It is difficult to explain why this works. I refer you to the textbook if you are sufficiently curious.

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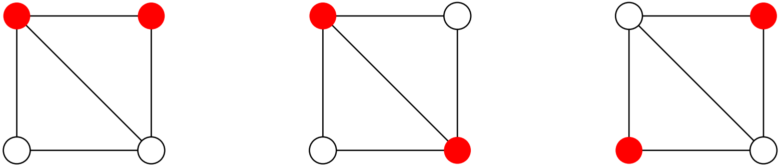

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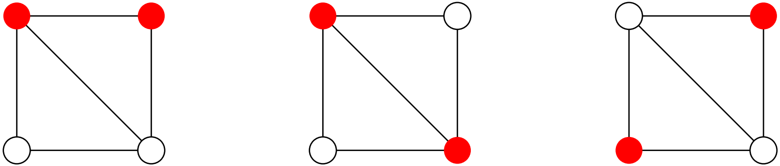

If there are three colors such as red, white and blue, we would substitute $x_1 = r + w + b$, $x_2 = r^2 + w^2 + b^2$, etc. The calculations become considerably more lengthy and this substitution for our small group is already seriously long. However, symbolic math software (such as Mathematica® or Maple®) can do it rapidly.

Without showing any details, for our current group:

$$P_G(r+b+w, r^2+b^2+w^2) = r^4 + w^4 + b^4 + 2r^3w + 2r^3b + 2w^3r + 2w^3b \\ + 2b^3r + 2b^3w + 3r^2w^2 + 3r^2b^2 + 3w^2b^2 + 4rwb^2 + 4rbw^2 + 4wbr^2$$

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(I did this by hand . . . I don't recommend doing that.)

Without showing any details, for our current group:

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(I did this by hand . . . I don't recommend doing that.) We can see from the last term (for example) there are 4 distinguishable ways to color the vertices where 2 are red, one white and one blue.

Without showing any details, for our current group:

$$P_G(r+b+w, r^2+b^2+w^2) = r^4 + w^4 + b^4 + 2r^3w + 2r^3b + 2w^3r + 2w^3b + 2b^3r + 2b^3w + 3r^2w^2 + 3r^2b^2 + 3w^2b^2 + 4rwb^2 + 4rbw^2 + 4wbr^2$$

(I did this by hand . . . I don't recommend doing that.) We can see from the last term (for example) there are 4 distinguishable ways to color the vertices where 2 are red, one white and one blue. Here they are: