Groups and Burnside's Theorem

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Here's a brief explanation of Burnside's Theorem for the group of rigid motions of an equilateral triangle.

$\mathscr{C} \backslash G$	(1)(2)(3)	(123)	(132)	(1)(23)	(2)(13)	(3)(12)		
WWW	1	1	1	1	1	1	6	6
RWW	1	0	0	1	0	0	2	
WRW	1	0	0	0	1	0	2	
WWR	1	0	0	0	0	1	2	6
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	8	2	2	4	4	4		24

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$(1)(2)(3)$	(123)	(132)	$(1)(23)$	$(2)(13)$	$(3)(12)$
2^3	2^1	2^1	2^2	2^2	2^2

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The group of rigid motions of the square is

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With 2 colors, the number of distinguishable colorings is

$$\frac{1}{8}\left(2^4 + 2^1 + 2^2 + 2^1 + 2^2 + 2^2 + 2^3 + 2^3\right) = \frac{48}{8} = 6.$$

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The group of rigid motions of the rectangle is

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With 2 colors, the number of distinguishable colorings is

$$\frac{1}{4}\left(2^4+2^2+2^2+2^2\right)=\frac{28}{4}=7.$$

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For the equilateral triangle, with three colors, the number of distinguishable colorings is

$$\frac{1}{6}\left(3^3+3^1+3^1+3^2+3^2+3^2\right)=\frac{60}{6}=10.$$

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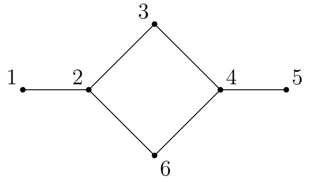

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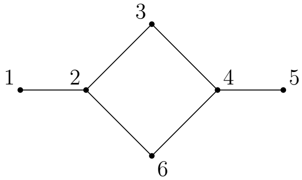

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The group of the above figure is
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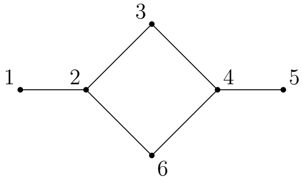

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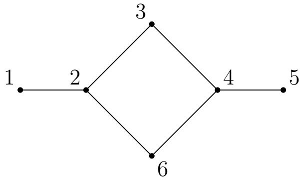

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With 5 colors: $\frac{1}{4}\left(5^6 + 5^3 + 5^4 + 5^5\right) = 4875$.

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One last example: the regular pentagon, whose group is

$$\{(1)(2)(3)(4)(5), (12345), (13524), (14253), (15432), (1)(25)(34), \\ (2)(13)(54), (3)(24)(15), (4)(35)(12), (5)(14)(23)\}.$$

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Exercise: for 2 colors, compute the number of distinguishable colorings of the vertices of a regular hexagon.