Correcting Errors

Daniel H. Luecking

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This is the generator matrix for a group code:

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(a) Write down the corresponding parity check matrix H:

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(b) For each of the received words $r = (0110\,010)$, $s = (0111\,000)$ and $t = (1001\,110)$: assume there is at most one error and use H to correct it where necessary (or explain why it cannot be corrected).

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For example to encode the messages $w = (01100)$ and $v = (01011)$ we multiply

$$wG = (01100\,0101) \quad \text{and} \quad vG = (01011\,1001)$$

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$$A = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{pmatrix} \implies A^{\mathrm{tr}} = \begin{pmatrix} 1 & 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 \end{pmatrix}$$

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and then appending an identity matrix to the right of A^{tr}:

$$H = \left(\begin{array}{ccccc|cccc} 1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 1 \end{array}\right)$$

Now the code words we send can be processed. We'll do this with the following words that might be received at the destination.

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To process r we compute Hr^{tr}. This will equal the sum of columns 2, 4, 8 and 9 of H:

$$Hr^{\mathrm{tr}} = \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

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This shows that r is a valid code word and the message being sent is (01010), obtained by dropping the last 4 bits appended by the encoding step.

To process $s = (11000\,1001)$ we compute Hs^{tr}. This will equal the sum of columns 1, 2, 6 and 9 of H:

$$Hs^{\mathrm{tr}} = \begin{pmatrix}1\\1\\0\\0\end{pmatrix} + \begin{pmatrix}0\\1\\1\\0\end{pmatrix} + \begin{pmatrix}1\\0\\0\\0\end{pmatrix} + \begin{pmatrix}0\\0\\0\\1\end{pmatrix} = \begin{pmatrix}0\\0\\1\\1\end{pmatrix}$$

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Finally, we get the original message by removing the last 4 bits from the corrected word: (11100)

To process $t = (00011\,1000)$ we compute Ht^{tr}. This will equal the sum of columns 4, 5 and 6 of H:

$$Ht^{\mathrm{tr}} = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 1 \\ 1 \end{pmatrix}$$

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Producing the original message would be nothing more than a guess.