Groups: Definition and Examples

Daniel H. Luecking
MASC

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There are is also rotation symmetry, meaning you can rotate the figure around a center point some number of degrees *without changing the figure*. The group involved here is not the figure, but rather the collection of *motions* that do not change the figure.

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- G3 For any a in G there is another element b such that $a * b = e = b * a$ (b is called the "inverse" of a. In examples, b can be written $-a$ or a^{-1}.)

Examples: groups associated with rings

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- The cancellation properties: if $ab = ac$ then $b = c$. This is because we can "multiply" both sides by a^{-1} to get $a^{-1}(ab) = a^{-1}(ac)$ then regroup to $(a^{-1}a)b = (a^{-1}a)c$. This is $eb = ec$ which says $b = c$. Similarly, if $ba = ca$ then $b = c$.

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- The identity is unique: if $ba = a$ write this as $ba = ea$ and then cancellation gives $b = e$.
- The inverse is unique: if b and c are inverses of a, so that $ab = e = ac$, use cancellation to get $b = c$.

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A further look at examples

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- The additive group of $\mathbb{Z}_2^n$ consists of strings of bits with length n. The operation is bitwise addition mod 2 (i.e., the bitwise $\textsc{xor}$).

More on permutations

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$$\begin{array}{r|cccc} x & 1 & 2 & 3 & 4 \\ \hline f(x) & 2 & 3 & 1 & 4 \end{array}$$

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defines a function f whose values are obtained by looking up x in the first row and reading off the value $f(x)$ below it. Notice that the second row is a permutation of the first row. Every different permutation will produce a different one-to-one function. This is in part why we simply call these functions permutations.

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If $\beta = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 4 & 1 & 2 \end{pmatrix}$ then $\alpha\beta = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 1 & 3 & 2 \end{pmatrix}$. The following diagrams can help to see this.

Permutation notation

It is traditional to name permutations with Greek letters. It is also traditional to write the composition of two permutation, say α and β, by $\alpha\beta$. This is interpreted as 'first α then β'. [In function notation this would be $\beta(\alpha(x))$ because we evaluate innermost parentheses first.]. We abbreviate the table on the previous page as $\alpha = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 1 & 4 \end{pmatrix}$, which defines α to be the same function as f.

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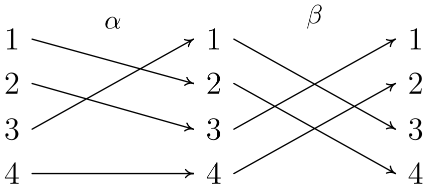
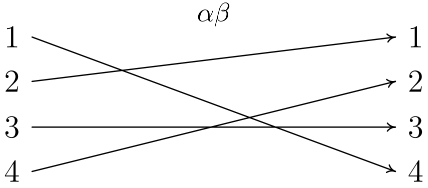
We can visualize functions using arrows. On the next slide we see arrows used to represent α and β, as well as $\alpha\beta$

Aids in composing permutation

Below $\alpha = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 1 & 4 \end{pmatrix}$, $\beta = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 4 & 1 & 2 \end{pmatrix}$, and $\alpha\beta = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 1 & 3 & 2 \end{pmatrix}$.





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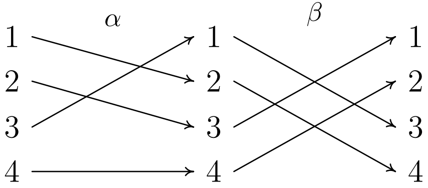

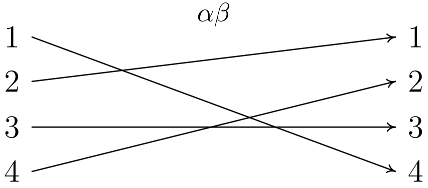

To get the second figure from the first, connect the head of an α-arrow to the tail of the β-arrow and straighten it out.

Aids in visualizing permutations

We can imagine permutations as representing motions. If $1, 2, 3,$ and 4 label four points in a plane (or in space) we can imagine a permutation as moving one point to another.

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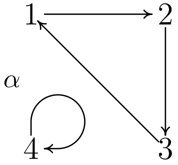

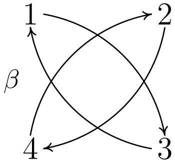

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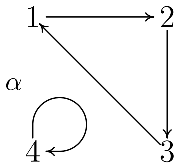

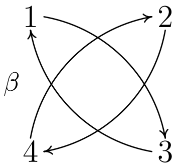

This can be used to compute $\alpha\alpha$. Just follow two arrows.

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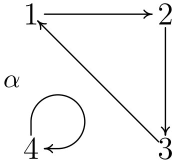

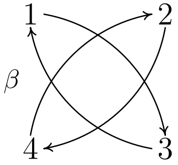

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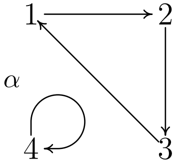
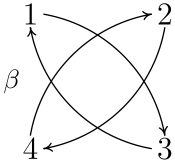
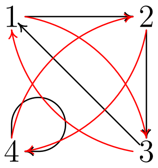




This can be used to compute $\alpha\alpha$. Just follow two arrows. For example starting from 1, the arrows go to 2 then 3, so we see that $\alpha\alpha$ moves 1 to 3. It is not so useful for computing $\alpha\beta$. One can do that by drawing both permutations in the same figure, with different colored arrows. See the figure on the next page.

Composing permutations as motions



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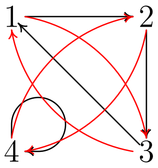

We can then follow the black arrow from 1 to 2 and then the red arrow from 2 to 4 to see that $\alpha\beta$ moves 1 to 4.

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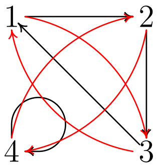
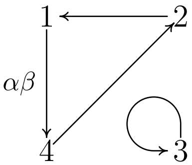


We can then follow the black arrow from 1 to 2 and then the red arrow from 2 to 4 to see that $\alpha\beta$ moves 1 to 4. This gives



From this we can read off $\alpha\beta = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 1 & 3 & 2 \end{pmatrix}$.