Combinatorics Review

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These are called arithmetic progressions. Any recurrence relation of the form $a_n = a_{n-1} + d, \;\; a_0 = c$ (where d and c are constants) has the solution $a_n = c + dn$.

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- $a_n = a_{n-1} + 4n,\ \ a_0 = 3$. Solution: $a_n = 3 + 4 + 8 + \cdots + 4n$, or $a_n = 3 + \sum_{k=1}^{n} 4k$.
- $a_n = a_{n-1} + 3^{n-1},\ \ a_0 = 2$. Solution: $a_n = 2 + 1 + 3 + 3^2 + \cdots + 3^{n-1}$, or $a_n = 2 + \sum_{k=1}^{n} 3^{k-1}$.

If a recurrence relation has the form $a_n = a_{n-1} + f(n)$ for some expression $f(n)$ and $a_0 = c$ is the initial condition, then the solution is $a_n = c + f(1) + f(2) + \cdots + f(n)$, or $a_n = c + \sum_{k=1}^{n} f(k)$.

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- $a_n = 4^n a_{n-1}$, $a_0 = 5$. Solution: $a_n = 5(4^1)(4^2)(4^3)\cdots(4^n)$.

If a recurrence relation has the form $a_n = f(n)a_{n-1}$ for some expression $f(n)$ and $a_0 = c$ is the initial condition, then the solution is $a_n = cf(1)f(2)f(3)\cdots f(n)$.

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- Nonhomogeneous: $a_n + ba_{n-1} + ca_{n-2} = f(n)$.

We solve the homogeneous case by first solving the characteristic equation: $r^2 + br + c = 0$. What we do next depends on what the roots are.

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Solving these gives $C_1 = 1/5$ and $C_2 = -1/5$, and the completed solution is

$$a_n = (1/5)3^n - (1/5)(-2)^n.$$

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Solving these gives $C_1 = 1$ and $C_2 = 2/5$, and the completed solution is

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Solving these gives $C_1 = 1 + i$ and $C_2 = 1 - i$, and the completed solution is

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An example: $a_n - a_{n-1} - 6a_{n-2} = (4)2^n$, $a_0 = 1$, $a_1 = 2$.

We solve it by first solving the corresponding homogeneous recurrence relation: $a_n - a_{n-1} - 6a_{n-2} = 0$. This gives the homogeneous solution:

$$a_n^{(h)} = C_1 3^n + C_2(-2)^n.$$

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Then we decide what a particular solution might look like. Since the right side is a constant times 2^n, we suppose one solution might be a constant times 2^n and so we put $a_n = A2^n$ into the recurrence relation and see what A will have to be:

$$A2^n - A2^{n-1} - 6A2^{n-2} = (4)2^n \implies -A = 4 \implies A = -4.$$

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Consider the recurrence relation $a_n - a_{n-1} - 2a_{n-2} = (3)2^n$, $a_0 = 0$, $a_1 = 1$. If we tried the same particular solution: $a_n = A2^n$ the same process would give us $0A = 3$, which is impossible for any A.

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shows that $A2^n$ is a solution of the homogeneous equation and so will always produce 0 and never $(3)2^n$. In cases like this we multiply the proposed solution $A2^n$ by n and try $a_n = An2^n$.

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Applying the initial conditions to this:

$$a_0 = 0 = C_1 + C_2 + 0 \quad \text{and} \quad a_1 = 1 = 2C_1 - C_2 + 4$$

we get $C_1 = -1$ and $C_2 = 1$ for a complete solution:

$$a_n = -2^n + (-1)^n + 2n2^n.$$

Computations in $\mathbb{Z}_n$

$\mathbb{Z}_n$ is the set $\{0, 1, \ldots, n-1\}$. We make it a ring by giving it two operations we call addition $+$ and multiplication $\cdot$, defined as follows.

$$m + k = (m + k \bmod n) \quad \text{and} \quad m \cdot k = (mk \bmod n)$$

For example, in $\mathbb{Z}_6$ we have the following operation tables:

$+$	0	1	2	3	4	5
0	0	1	2	3	4	5
1	1	2	3	4	5	0
2	2	3	4	5	0	1
3	3	4	5	0	1	2
4	4	5	0	1	2	3
5	5	0	1	2	3	4

$\cdot$	0	1	2	3	4	5
0	0	0	0	0	0	0
1	0	1	2	3	4	5
2	0	2	4	0	2	4
3	0	3	0	3	0	3
4	0	4	2	0	4	2
5	0	5	4	3	2	1

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In $\mathbb{Z}_6$ only the elements 1 and 5 are units with $1^{-1} = 1$ and $5^{-1} = 5$. In $\mathbb{Z}_7$, all the elements except 0 are units and

$$1^{-1} = 1, \quad 2^{-1} = 4, \quad 3^{-1} = 5, \quad 4^{-1} = 2, \quad 5^{-1} = 3, \quad 6^{-1} = 6.$$

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For $\mathbb{Z}_{2100}$ there are $2100 - 480 - 1 = 1619$ proper zero divisors. You will be expected to do this for any $\mathbb{Z}_n$ if I give you the factorization of n.

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where $n = 409$, $k = 101$, $r_1 = 5$ and $r_2 = 1$.

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Solving for r_2: $r_2 = 81k - 20n$. Putting the actual values back:
$1 = 81(101) - 20(409)$. This tells us that, in the ring $\mathbb{Z}_{409}$, $1 = 81 \cdot 101$ and so $(101)^{-1} = 81$.