The Rings $\mathbb{Z}_n$

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March 11, 2024

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we can eliminate r_1 by inserting its value $(n - 7k)$ in the second equation:

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This tells us that $64 \cdot 100 = 1$ in $\mathbb{Z}_{711}$ so, $100^{-1} = 64$.

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This tells us that $176 \cdot 101 = 1$ in $\mathbb{Z}_{711}$ so, $101^{-1} = 176$.

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If we know the prime factorization of n there is a relatively simple formula for $\phi(n)$. The first thing we remark is that if d evenly divides both k and n, then any prime factor of d also does so.

So, if we want to eliminate numbers with $\gcd(n, k) > 1$ we only need to test primes.
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$$S_1 = n\left(\frac{1}{p_1} + \frac{1}{p_2} + \frac{1}{p_3}\right).$$

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Putting these together

$$\begin{aligned} N(\overline{c_1}\,\overline{c_2}\,\overline{c_3}) &= n - n\left(\frac{1}{p_1} + \frac{1}{p_2} + \frac{1}{p_3}\right) + n\left(\frac{1}{p_1p_2} + \frac{1}{p_1p_3} + \frac{1}{p_2p_3}\right) - n\left(\frac{1}{p_1p_2p_3}\right) \\ &= n\left(1 - \frac{1}{p_1} - \frac{1}{p_2} - \frac{1}{p_3} + \frac{1}{p_1p_2} + \frac{1}{p_1p_3} + \frac{1}{p_2p_3} - \frac{1}{p_1p_2p_3}\right) \\ &= n\left(1 - \frac{1}{p_1}\right)\left(1 - \frac{1}{p_2}\right)\left(1 - \frac{1}{p_3}\right) \end{aligned}$$

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In particular, if p is a prime number then $\phi(p) = p(1-1/p) = p-1$, $\phi(p^2) = p^2(1-1/p) = p(p-1)$, etc.

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$$\begin{aligned}\phi(90) &= 90\left(1-\frac{1}{2}\right)\left(1-\frac{1}{3}\right)\left(1-\frac{1}{5}\right)\\ &= 2(3^2)5\left(\frac{1}{2}\right)\left(\frac{2}{3}\right)\left(\frac{4}{5}\right)\\ &= 3(1)(2)(4) = 24\end{aligned}$$

Another example: find $\phi(2200)$. Since $2200 = 2^3 5^2 11$ we get

$$\begin{aligned}\phi(2200) &= 2^3 5^2 11 \left(\frac{1}{2}\right)\left(\frac{4}{5}\right)\left(\frac{10}{11}\right) \\ &= 2^2 5(1)(4)(10) = 800.\end{aligned}$$

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Since $\phi(90) = 24$, the ring $\mathbb{Z}_{90}$ has 24 units and $90 - 24 - 1 = 65$ proper zero divisors.

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The ring $\mathbb{Z}_{911}$ has $\phi(911) = 910$ units and $911 - 910 - 1 = 0$ proper zero divisors.
(911 is prime so $\phi(911) = 911\left(1 - \frac{1}{911}\right) = 911 - 1 = 910$.)

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Proving the theorem is maybe a little tricky but not particularly long. The first two conditions are the definition of h being a *homomorphism*.

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- $h(0) = 0$ and $h(-x) = -h(x)$ for all $x \in R$.

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- h is a one-to-one correspondence between proper zero divisors in R and the proper zero divisors in Y.
- If R has a unity 1, so does Y and $h(1)$ is the unity of Y. Also if $x \in R$ is a unit then $h(x^{-1}) = h(x)^{-1}$.
- $h(0) = 0$ and $h(-x) = -h(x)$ for all $x \in R$.

Theorem

If $n = lm$ then h defined by $h(k) = (k \bmod l, k \bmod m)$ is a homomorphism from $\mathbb{Z}_n$ to $\mathbb{Z}_l \times \mathbb{Z}_m$.

The 'Chinese Remainder' Theorem

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A special case is $n = pq$ where p and q are different primes.