Recurrence Relations

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Right side	form of particular solution
5	A
$(5)3^n$	$A3^n$
$n-2$	$An+B$
$2n^3+3n$	An^3+Bn^2+Cn+D
$3n2^n$	$(An+B)2^n$
$(n^3+2)5^n$	$(An^3+Bn^2+Cn+D)5^n$

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The table leads us to try a particular solution of the form $a_n = An^2 + Bn + C$.

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$$\begin{aligned} a_{n-1} &= A(n-1)^3 + B(n-1)^2 + C(n-1) \\ &= [An^3 - 3An^2 + 3An - A] + [Bn^2 - 2Bn + B] + [Cn - C] \\ &= An^3 + (-3A + B)n^2 + (3A - 2B + C)n + (-A + B - C) \end{aligned}$$

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This leads to

$$\begin{aligned}
3A \qquad\qquad &= 1 & A &= 1/3 \\
-3A + 2B \quad\; &= 0 & B &= 1/2 \\
A - \;\; B + C &= 0 & C &= 1/6
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So, $a_n^{(p)} = (1/3)n^3 + (1/2)n^2 + (1/6)n$

Or simplified: $a_n^{(p)} = (2n^3 + 3n^2 + n)/6$. The general solution is $a_n = C_1 + (2n^3 + 3n^2 + n)/6$ and the initial condition gives us $C_1 = 1$.

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and then adding all the equations together, starting with $n = 1$:

$$(a_1 x + a_2 x^2 + a_3 x^3 + \cdots) - (a_0 x + a_1 x^2 + a_2 x^3 + \cdots) = \\ 2(1+1)x^1 + 2(2+1)x^2 + 2(3+1)x^3 + \cdots$$

Or, more concisely

$$\sum_{n=1}^{\infty} a_n x^n - \sum_{n=1}^{\infty} a_{n-1} x^n = 2 \sum_{n=1}^{\infty} (n+1) x^n. \tag{$*$}$$

Or, more concisely

$$\sum_{n=1}^{\infty} a_n x^n - \sum_{n=1}^{\infty} a_{n-1} x^n = 2 \sum_{n=1}^{\infty} (n+1) x^n. \qquad (*)$$

If $F(x) = \sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots$ is the generating function, then The first sum in $(*)$ is $F(x) - a_0 = F(x) - 1$.

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$$2(2x + 3x^2 + 4x^3 + \cdots) = 2\left(\frac{1}{(1-x)^2} - 1\right).$$

With these substitutions

$$\begin{aligned} F(x) - 1 - xF(x) &= \frac{2}{(1-x)^2} - 2 \\ (1-x)F(x) &= \frac{2}{(1-x)^2} - 1 \\ F(x) &= \frac{2}{(1-x)^3} - \frac{1}{1-x} \end{aligned}$$

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From the formulas in Chapter 9, we get

$$F(x) = 2\sum_{n=0}^{\infty} \binom{n+2}{n} x^n - \sum_{n=0}^{\infty} x^n = \sum_{n=0}^{\infty} \left[2\binom{n+2}{n} - 1 \right] x^n$$

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a_n = 2\binom{n+2}{n} - 1 = (n+2)(n+1) - 1 = n^2 + 3n + 1
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A second order example

$$\begin{aligned}
a_n - 4a_{n-1} + 4a_{n-2} = 2^n, \quad n \geq 2 \\
a_0 = 1,\ a_1 = 3
\end{aligned}$$

Multiply $a_n - 4a_{n-1} + 4a_{n-2} = 2^n$ by x^n:

$$a_n x^n - 4a_{n-1}x^n + 4a_{n-2}x^n = 2^n x^n, \quad n \geq 2$$

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the second is

$$\sum_{n=2}^{\infty} a_{n-1}x^n = x\sum_{n=2}^{\infty} a_{n-1}x^{n-1} = x(a_1 x + a_2 x^2 + a_3 x^3 + \cdots) = x(F(x) - a_0)$$

the third is

$$\sum_{n=2}^{\infty} a_{n-2}x^n = x^2 \sum_{n=2}^{\infty} a_{n-2}x^{n-2} = x^2(a_0 + a_1x + a_2x^2 + \cdots) = x^2F(x)$$

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Thus the equation for $F(x)$ is

$$[F(x) - a_0 - a_1x] - 4[x(F(x) - a_0)] + 4[x^2F(x)] = \sum_{n=2}^{\infty}(2x)^n$$

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Finally, with $a_0 = 1$ and $a_1 = 3$ we get

$$F(x) - 1 - 3x - 4x(F(x) - 1) + 4x^2F(x) = \frac{(2x)^2}{1-2x}.$$

Or

$$
\begin{aligned}
(1-4x+4x^2)F(x)-1-3x+4x &= \frac{4x^2}{1-2x} \\
(1-4x+4x^2)F(x) &= 1-x+\frac{4x^2}{1-2x} \\
F(x) &= \frac{1-x+4x^2/(1-2x)}{1-4x+4x^2} \\
F(x) &= \frac{1-3x+6x^2}{(1-2x)^3}
\end{aligned}
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If the sequence a_n satisfies the following recurrence relation and initial condition, find its generating function without solving the recurrence relation.

$$a_n - 5a_{n-1} + 6a_{n-2} = 5, \quad n \geq 2$$
$$a_0 = 1,\ a_1 = 5.$$

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Equation for the generating function:

$$\begin{aligned}
F(x) - 1 - 5x - 5x(F(x) - 1) + 6x^2 F(x) &= 5 \sum_{n=2}^{\infty} x^n \\
F(x) - 1 - 5x - 5xF(x) + 5x + 6x^2 F(x) &= \frac{5x^2}{1-x} \\
(1 - 5x + 6x^2) F(x) &= 1 + \frac{5x^2}{1-x} \\
F(x) &= \frac{1 + 5x^2/(1-x)}{1 - 5x + 6x^2} \\
F(x) &= \frac{1 - x + 5x^2}{(1-x)(1-2x)(1-3x)}
\end{aligned}$$

If the sequence a_n satisfies the following recurrence relation and initial condition, find its generating function without solving the recurrence relation.

$$\begin{aligned} a_n - 2a_{n-1} + 5a_{n-2} &= (-1)^n, \quad n \geq 2 \\ a_0 = 1,\ a_1 &= 1. \end{aligned}$$

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$$F(x) - 1 - x - 2xF(x) + 2x + 5x^2F(x) = \frac{x^2}{1+x}$$
$$(1 - 2x + 5x^2)F(x) = 1 - x + \frac{x^2}{1+x}$$
$$F(x) = \frac{1 - x + x^2/(1+x)}{1 - 2x + 5x^2}$$
$$F(x) = \frac{1}{(1+x)(1-2x+5x^2)}$$

General solutions

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Equation for the generating function:

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$$F(x) - a_0 + (2a_0 - a_1)x - 2xF(x) + 5x^2 F(x) = \frac{x^2}{1+x}$$

$$(1 - 2x + 5x^2)F(x) = a_0 - (2a_0 - a_1)x + \frac{x^2}{1+x}$$

$$F(x) = \frac{a_0 - (2a_0 - a_1)x + x^2/(1+x)}{1 - 2x + 5x^2}$$

Third order equation:

$$a_n - 5a_{n-1} + 2a_{n-2} + 2a_{n-3} = 1, \quad n \geq 3$$
$$a_0 = 1,\ a_1 = 1,\ a_2 = 3.$$

Third order equation:

$$a_n - 5a_{n-1} + 2a_{n-2} + 2a_{n-3} = 1, \quad n \geq 3$$
$$a_0 = 1,\ a_1 = 1,\ a_2 = 3.$$

The sum goes from $n = 3$ to ∞, so we get for the generating function:

$$F(x) - 1 - x - 3x^2 - 5x(F(x) - 1 - x) + 2x^2(F(x) - 1) + 2x^3 F(x) = \sum_{n=3}^{\infty} x^n.$$

$$F(x) - 1 + 4x - 5xF(x) + 2x^2 F(x) + 2x^3 F(x) = \frac{x^3}{1-x}$$

$$(1 - 5x + 2x^2 + 2x^3)F(x) = 1 - 4x + \frac{x^3}{1-x}$$

$$F(x) = \frac{1 - 4x + x^3/(1-x)}{1 - 5x + 2x^2 + 2x^3} = \frac{1 - 5x + 4x^2}{(1-x)^2(1 - 4x - 2x^2)}$$