# Recurrence Relations

Daniel H. Luecking

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Here is a different kind of right side for a nonhomogeneous recurrence relation:

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Putting $a_n = An + B$, $a_{n-1} = A(n-1) + B = An - A + B$ and $a_{n-2} = A(n-2) + B = An - 2A + B$ into the recurrence relation, we get

$$\begin{aligned} An + B - (An - A + B) - 2(An - 2A + B) &= 4n \\ An + B - An + A - B - 2An + 4A - 2B &= 4n \\ -2An - 2B + 5A &= 4n \end{aligned}$$

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This gives us a particular solution $a_n^{(p)} = -2n - 5$. We add this to the homogeneous solution $a_n^{(h)} = C_1(-1)^n + C_2 2^n$ to get our general solution

$$a_n = C_1(-1)^n + C_2 2^n - 2n - 5.$$

With $a_n = C_1(-1)^n + C_2 2^n - 2n - 5$ as our general solution and initial conditions $a_0 = 1$, $a_1 = 2$, the system of equations for $C_1$ and $C_2$ is

$$\begin{aligned} C_1 + \ \ C_2 - 5 &= 1 \\ -C_1 + 2C_2 - 7 &= 2 \end{aligned}$$

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The general rule is that if the right side is a polynomial in the variable $n$, then $a_n^{(p)}$ will likely also be a polynomial of the same degree, but may have terms that the right side is missing.

For example:

| Right side | form of particular solution |
|---|---|
| $5$ | $A$ |
| $(5)3^n$ | $A3^n$ |
| $n-2$ | $An+B$ |
| $2n^3+3n$ | $An^3+Bn^2+Cn+D$ |
| $3n2^n$ | $(An+B)2^n$ |
| $(n^3+2)5^n$ | $(An^3+Bn^2+Cn+D)5^n$ |

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An exceptional case: If the right side is similar to the homogeneous solution, then a simple application of this method will often fail.

Here is an example of that:

$$a_n - a_{n-1} - 2a_{n-2} = 2^n, \quad n \geq 2$$

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The homogeneous solution is $a_n^{(h)} = C_1(-1)^n + C_2 2^n$. Here's what happens if we try $a_n = A2^n$:

$$\begin{aligned} A2^n - A2^{n-1} - 2A2^{n-2} &= 2^n \\ (A - A2^{-1} - 2A2^{-2})2^n &= 2^n \\ (A - A/2 - 2A/4) &= 1 \\ 0 &= 1 \end{aligned}$$

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This is something we could have predicted: Since $A2^n$ is a solution of the homogeneous equation, putting it in the recurrence relation will produce a zero on the left side.

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gives us

$$\begin{aligned}
An2^n - A(n-1)2^{n-1} - 2A(n-2)2^{n-2} &= 2^n \\
[An - A(n-1)2^{-1} - 2A(n-2)2^{-2}]2^n &= 2^n \\
An - (A/2)(n-1) - (A/2)(n-2) &= 1 \\
An - (A/2)n + A/2 - (A/2)n + A &= 1 \\
3A/2 &= 1
\end{aligned}$$

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so $A = 2/3$ and $a_n^{(p)} = (2/3)n2^n$, general solution $a_n = C_1(-1)^n + C_2 2^n + (2/3)n2^n$.

Lets throw in some initial conditions $a_0 = 0$, $a_1 = 0$ to get

$$\begin{aligned} C_1 + \ \ C_2 + \quad 0 &= 0 \\ -C_1 + 2C_2 + 4/3 &= 0 \end{aligned}$$

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giving $C_1 = 4/9$ and $C_2 = -4/9$ for a completed solution
$a_n = (4/9)(-1)^n + (-4/9)2^n + (2/3)n2^n$.

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There is a general rule. If any term of the proposed particular solution is a solution of the homogeneous equation, then multiply by $n$. If the new proposed solution has the same problem, multiply by $n$ again.

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The homogeneous solution is $a_n^{(h)} = C_1 2^n + C_2 n2^n$. The form of the solution given by the table is $(An + B)2^n = An2^n + B2^n$.

Both terms of $An2^n + B2^n$ are solutions of the homogeneous equation so we multiply by $n$ and consider $An^2 2^n + Bn2^n$.

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One final point: we can also handle right sides that are sums of things in the table.

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One final point: we can also handle right sides that are sums of things in the table. The basis for this is the following general principle:

**Theorem**

*If you find particular solutions for*

$$a_n + ba_{n-1} + ca_{n-2} = f(n) \quad \textit{and for} \quad a_n + ba_{n-1} + ca_{n-2} = g(n)$$

*then their sum will be a particular solution of*

$$a_n + ba_{n-1} + ca_{n-2} = f(n) + g(n)$$

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This means that for

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a particular solution is $a_n^{(p)} = -2n - 5 + (2/3)n2^n$. And of course the general solution is
$a_n = a_n^{(h)} + a_n^{(p)} = C_1(-1)^n + C_2 2^n - 2n - 5 + (2/3)n2^n$.

A completely worked out example:

$$a_n - 3a_{n-1} + 2a_{n-2} = 4 + (2)3^n, \quad n \geq 2$$
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Putting $a_n = A$ won't work because a constant is a solution of the homogeneous equation. Thus we use $An$. This gives

$$An - 3A(n-1) + 2A(n-2) = 4$$
$$-A = 4$$

so $A = -4$ and $An = -4n$ is the corresponding particular solution.

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Adding these gives $a_n^{(p)} = -4n + (9)3^n$. and so the general solution of

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is $a_n = C_1 + C_2 2^n - 4n + (9)3^n$. The equations for $C_1$ and $C_2$ are

$$C_1 + \ \ C_2 + \ \ 9 = 0$$
$$C_1 + 2C_2 + 23 = 1$$

giving $C_1 = 4$, $C_2 = -13$

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giving $C_1 = 4$, $C_2 = -13$ so the completed solution is $a_n = 4 - (13)2^n - 4n + (9)3^n$.