Recurrence Relations

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can be rewritten

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Thus $\alpha + \beta i = \rho(\cos \theta + i \sin \theta)$. A famous theorem due to Euler says that

$$(\alpha + \beta i)^n = \rho^n (\cos \theta + i \sin \theta)^n = \rho^n (\cos(n\theta) + i \sin(n\theta))$$

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$$a_n = \rho^n [C_1 \cos(n\theta) + C_2 \sin(n\theta)]$$

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In our example where $r = 2 \pm 2i$ we get $\rho = \sqrt{2^2 + 2^2} = 2\sqrt{2}$ and $\theta = \cos^{-1}(1/\sqrt{2}) = 45^\circ$ (or $\pi/4$ radians).

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$$a_n = (2\sqrt{2})^n[C_1 \cos(45n) + C_2 \sin(45n)]$$

The initial conditions of that example ($a_0 = 2$, $a_1 = 12$) give the following equation for C_1 and C_2 (note that $\cos 0 = 1$, $\sin 0 = 0$, $\cos 45^\circ = \sqrt{2}/2$ and $\sin 45^\circ = \sqrt{2}/2$)

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Another example with complex roots, which I will process both ways:

$$a_n - 2a_{n-1} + 5a_{n-2} = 0, \quad n \geq 0$$
$$a_0 = 0,\ a_1 = 3$$

Characteristic equation $r^2 - 2r + 5 = 0$, with roots $1 \pm 2i$.

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with solution $C_1 = -3i/4$, $C_2 = 3i/4$.

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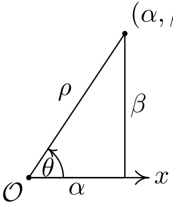
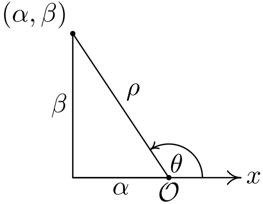
Something to note about the trigonometric function method: These two pictures represent the ρ and θ for two examples. The first is where α is positive and so the point (α, β) is to the right of the y axis The second is where α is negative and so the point (α, β) is to the left of the y axis





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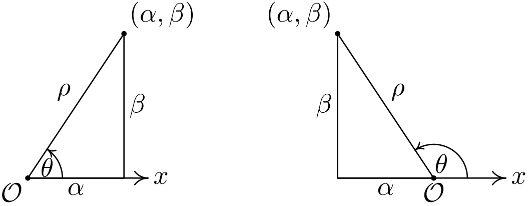

Because $\rho \cos\theta = \alpha$ and $\rho \sin\theta = \beta$ always hold, the initial conditions for $a_n = \rho^n[C_1 \cos(n\theta) + C_2 \sin(n\theta)]$ always simplify to

$$\begin{aligned} C_1 &= a_0 \\ \alpha C_1 + \beta C_2 &= a_1 \end{aligned}$$

Note that $\theta = \cos^{-1}(\alpha/\rho)$ always gives an angle between 0 and 180, while seemingly equivalent formulas like $\sin^{-1}(\beta/\rho)$ and $\tan^{-1}(\beta/\alpha)$ will be wrong when $\theta > 90^\circ$.

Nonhomogeneous recurrence relations

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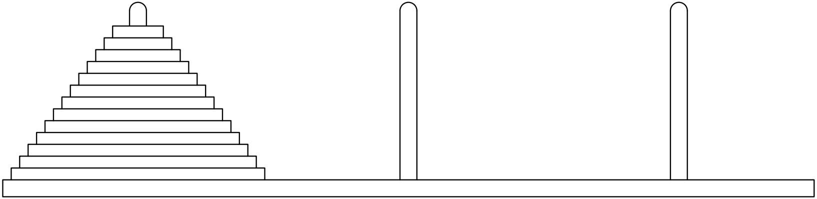
Here is a picture of a possible starting point with 13 disks:



The goal is to move all the disks from the first pole to the last.

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- One disk at a time may be moved from the top of one stack and placed on any of the other two poles.
- It is not allowed to place a disk on top of a smaller disk.
- The game ends when all the disks are on the third pole

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How many moves does it take? Let a_n be the number of moves for n disks. The above process requires a_{n-1} moves to get the top $n-1$ disks from pole 1 to 2, then 1 move to get the bottom disk from 1 to 3 and then a_{n-1} moves to get the other $n-1$ disks from 2 to 3.

That is

$$\begin{aligned} a_n &= 2a_{n-1}+1, \quad n \geq 1 \\ a_0 &= 0. \end{aligned}$$

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This has characteristic equation $r - 2 = 0$ with the root $r = 2$ and so the general solution is $a_n = C_1 2^n$. However, this doesn't solve the original nonhomogeneous equation.

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This exemplifies a general rule: *to find the general solution of a nonhomogeneous recurrence relation, just find one solution and add it to the general solution of the associated homogeneous recurrence relation.*

To see how we can use this, we examine our recurrence relation:

$$a_n - 2a_{n-1} = 1$$

and remark that since the right side is constant, we need the left side to be constant, so perhaps letting a_n be a constant will give us a solution.

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Following our previous discussion, we know that all other solutions have the form $a_n = C_1 2^n - 1$. Now that we have the general solution, we can impose the initial condition to find C_1:

$$C_1 2^0 - 1 = 0 \quad \text{or} \quad C_1 = 1.$$

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This actually works surprisingly often. As an example, lets work through the following. For the moment, lets not even mention initial conditions.

$$a_n - 4a_{n-1} + 3a_{n-2} = (3)2^n, \quad n \geq 2$$

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This actually works surprisingly often. As an example, lets work through the following. For the moment, lets not even mention initial conditions.

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To keep straight which expressions solve which equations, we'll call this the *homogeneous solution* and denote it by $a_n^{(h)}$.

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Thus, we should set $a_n = A2^n$ and run this through the equation to see what A should be.

If $a_n = A2^n$ then $a_{n-1} = A2^{n-1}$ and $a_{n-2} = A2^{n-2}$. Putting these in $a_n - 4a_{n-1} + 3a_{n-2} = (3)2^n$ gives

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Step 3: Add the 2 parts together to get the general solution. The general solution is $a_n = a_n^{(h)} + a_n^{(p)} = C_1 3^n + C_2 - (12)2^n$.

When initial conditions are present there is a final step.

Step 4: Use the initial conditions to find, and then fill in, the constants.

Lets illustrate this for the current problem. Here are some simple initial conditions

$$a_n - 4a_{n-1} + 3a_{n-2} = (3)2^n, \quad n \geq 2$$
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We obtain C_1 and C_2 by putting $n = 0$ and 1 into our general solution $a_n = C_1 3^n + C_2 - (12)2^n$:

$$C_1 + C_2 - 12 = 0$$
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Therefore, the completed solution is $a_n = (6)3^n + 6 - (12)2^n$.