Recurrence Relations

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February 21, 2024

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It should be clear that $a_1 = a_0 + 1 = 0 + 1 = 1$, $a_2 = a_1 + 1 = 1 + 1 = 2$, and so on. We can guess that $a_n = n$ for all n.

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Putting these 2 into the recurrence relation

$a_n = a_{n-1} + 1$ gives
$n = n - 1 + 1$, which is true for all $n \geq 1$.

Another example:

$$\begin{aligned} a_n &= 2a_{n-1} + 3, \quad n \geq 1 \\ a_0 &= 2. \end{aligned}$$

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Then check the the recurrence relation. Since $a_{n-1} = 5(2^{n-1}) - 3$, we have to see if

$5(2^n) - 3 = 2(5(2^{n-1}) - 3) + 3$.

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As we learn techniques for solving recurrence relations, keep in mind that a single recurrence relation like $a_n = 2a_{n-1} + 3$ is, in reality, an infinite sequence of equations.

That infinite sequence of equations is, counting the initial condition,

$$
\begin{aligned}
a_0 &= 2 \\
a_1 &= 2a_0 + 3 \\
a_2 &= 2a_1 + 3 \\
a_3 &= 2a_2 + 3 \\
&\vdots
\end{aligned}
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Every recurrence can be programmed into a loop that will generate some of the values. For example

```
numeric a[];
a[0] = 2;
for n = 1 upto 1000:
  a[n] := 2*a[n-1] + 3;
endfor
```

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The formula is actually $a_n = 2 \cdot n!$. Because putting this and $a_{n-1} = 2(n-1)!$ into the recurrence relation gives:

$$2(n!) = n(2(n-1)!),$$

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$$\begin{aligned} a_n &= a_{n-1} + a_{n-2}, \quad n \geq 2 && \text{order } 2 \\ a_{n+4} &= a_{n+3}a_{n+2} + 2a_n, \quad n \geq 0 && \text{order } 4 \\ a_n &= \textstyle\sum_{j=0}^{n-1} a_j, \quad n \geq 1 && \text{order } \infty \end{aligned}$$

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then the last relation above can be written $s_n - s_{n-1} = s_{n-1}$, which has order 1.

Some special first-order recurrence relations

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is an *arithmetic progression* with solution $a_n = c + dn$. Note that $a_n - a_{n-1} = d$, and so an arithmetic progression is one where the difference between successive terms is constant.

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We can solve it as follows: imagine all the equations between the first and the nth:

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Now imagine adding these together...

$$a_1 + a_2 + \cdots + a_n = a_0 + a_1 + \cdots + a_{n-1} + 1 + 2 + \cdots + n$$

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Now cancel common terms from both sides (a_1 through a_{n-1}) to get $a_n = a_0 + (1 + 2 + \cdots + n) = 1 + \frac{n(n+1)}{2}$.

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In general, a recurrence relation of the form

$$a_n = a_{n-1} + f(n), \quad n \geq 1$$

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Can be solved similarly: add the following

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to get $a_1 + a_2 + \cdots + a_n = a_0 + a_1 + \cdots + a_{n-1} + \sum_{j=1}^{n} f(j)$. Then cancel to get $a_n = a_0 + \sum_{j=1}^{n} f(j)$. Then fill in the initial condition.

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$$\begin{aligned} a_1 &= 2^0 a_0 \\ a_2 &= 2^1 a_1 \\ &\vdots \\ a_n &= 2^{n-1} a_{n-1} \end{aligned}$$

Now imagine *multiplying* these together. . .

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$$a_n = f(n) a_{n-1}, \quad n \geq 1$$

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A couple more examples

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a_n &= a_{n-1} + 3^n, \quad n \geq 1 \\
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Sometimes we can simplify these further (and sometimes we can't). I will never expect you to simplify such answers.