Combinatorics Review

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And so the number of derangements is

$$\begin{aligned} d_n = N - S_1 + S_2 - \cdots \pm S_n &= n! - \frac{n!}{1!} + \frac{n!}{2!} - \frac{n!}{3!} + \cdots \pm \frac{n!}{n!} \\ &= n!\left(1 - \frac{1}{1!} + \frac{1}{2!} - \cdots \pm \frac{1}{n!}\right) \end{aligned}$$

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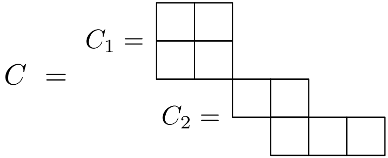
The rook numbers of a given chessboard can be placed in a polynomial called the *rook polynomial*, $r(C,x)$. Thus,

$$r(C,x) = \sum_{k=0}^{\infty} r_k(C)x^k.$$

We have two formulas for finding rook polynomals. The product formula only applies when the chessboard has the following format:



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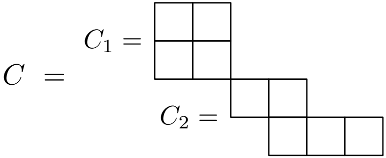

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In that case we have the formula $r(C, x) = r(C_1, x)r(C_2, x)$. In the above example, we get

$$r(C, x) = (1 + 4x + 2x^2)(1 + 5x + 5x^2) = 1 + 9x + 27x^2 + 30x^3 + 10x^4.$$

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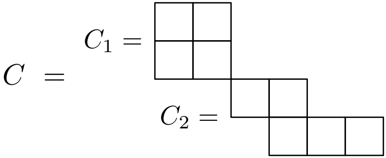

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Note that r_0 is always 1 and r_1 is always the number of squares.

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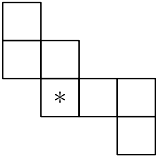
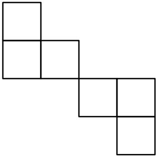
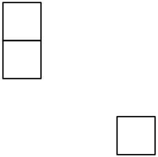
Our second formula applies to any chessboard. For example, chessboard C below:

$C =$  $C_e =$  $C_s =$ 

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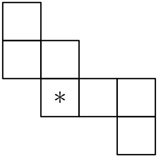
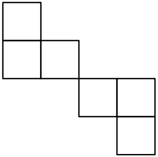
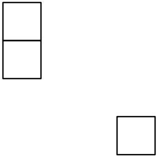
$C =$ 

$C_e =$ 
$C_s =$ 

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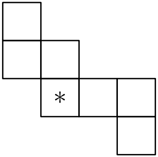
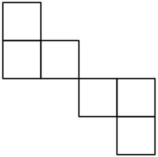
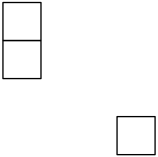
C =  C_e =  C_s = 

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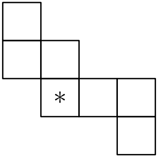
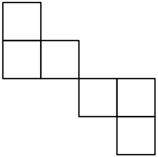
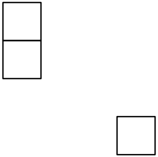
$C =$ 

$C_e =$ 
$C_s =$ 

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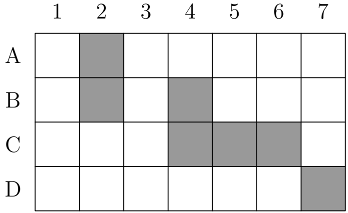
The formula is $r(C, x) = r(C_e, x) + x \cdot r(C_s, x)$. In this example, we get

$$\begin{aligned} r(C, x) &= (1 + 3x + x^2)(1 + 3x + x^2) + x(1 + 2x)(1 + x) \\ &= 1 + 7x + 14x^2 + 8x^3 + x^4 \end{aligned}$$

We apply rook polynomials to problems like the following:



This diagram represents the possibilities for seating 4 people in 7 seats. The shaded squares correspond to forbidden seats. We want to compute the number of ways to seat them all without putting anyone in a forbidden seat.

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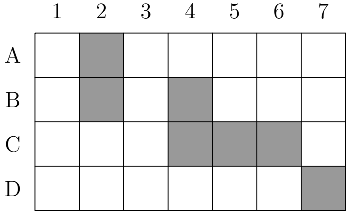

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If we ignore which assignments are forbidden, there are $P(7,4)$ ways. This is the N of an inclusion-excusion problem.

We use the conditions $c_j =$ 'the jth person is seated in a forbidden seat'. We discovered that

$$S_1 = r_1 P(6,3), \quad S_2 = r_2 P(5,2), \quad S_3 = r_3 P(4,1), \quad S_4 = r_4 P(3,0)$$

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where the r_k are the rook numbers for the chessboard of shaded squares. We can compute the rook polynomial to be $1 + 7x + 15x^2 + 11x^3 + 2x^4$. So that the number of ways to seat these 4 people respecting the forbidden seating is

$$\begin{aligned} N - S_1 + S_2 - S_3 + S_4 & \\ &= P(7,4) - 7P(6,3) + 15P(5,2) - 11P(4,1) + 2P(,0) \\ &= \frac{7!}{3!} - 7\frac{6!}{3!} + 15\frac{5!}{3!} - 11\frac{4!}{3!} + 2\frac{3!}{3!} \end{aligned}$$

Generating functions

We seek the number of solution to equations like the following

$b_1 + b_2 + b_3 = n$

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We first find the generating functions for the one-variable problems. This is entirely determined by the conditions. Thus,

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for b_3 the generating function is $1 + x + x^2 + \cdots x^{29} = \dfrac{1 - x^{30}}{1 - x}$

So the generating function for the complete problem is the product of these:

$$F(x) = \frac{x^2 - x^{32}}{1-x}\,\frac{x^8}{1-x}\,\frac{1-x^{20}}{1-x} = \frac{x^{10} - 2x^{40} + x^{70}}{(1-x)^3}$$

The meaning of ‘generating function’ is that it encodes the sequence in question (number of solutions for different n in this case) as the coefficients of x^n. We find the number of solutions for any value of n by finding x^n in $F(x)$ and reading off the number.

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Let’s take $n = 90$. First we rewrite $F(x)$:

$$F(x) = (x^{10} - 2x^{40} + x^{70}) \sum_{j=0}^{\infty} \binom{j+2}{j} x^j$$

we find x^{90} in these three terms:

$$x^{10} \binom{82}{80} x^{80} - 2x^{40} \binom{52}{50} x^{50} + x^{70} \binom{22}{20} x^{20}.$$

So the number of solutions is $\binom{82}{80} - 2\binom{52}{50} + \binom{22}{20}$.