Generating Functions, cont.

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February 12, 2024

Introducing the main problem

How many solutions does the following equation have if all the variables are required to be integers satisfying the stated conditions.

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\begin{aligned}
y_1 + y_2 + y_3 + y_4 &= n \\
0 &\le y_1 \\
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We turn to the next variable $y_2 = j$, $y_2 \geq 5$. Because of the condition, this has no solutions for $j < 5$. That is, if b_j is the number of solutions, then $b_0 = 0$, $b_1 = 0$, $b_2 = 0$, $b_3 = 0$ and $b_4 = 0$.

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$$x^5 + x^6 + x^7 + \cdots = x^5(1 + x + x^2 + \cdots) = \frac{x^5}{1 - x}.$$

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Multiplying the 4 generating functions together we get the grand generating function:

$$G(x) = \frac{1}{1-x} \frac{x^5}{1-x} \frac{1 - x^{20}}{1-x} \frac{x^{10}(1 - x^{20})}{1-x} = \frac{x^{15} - 2x^{35} + x^{55}}{(1-x)^4}$$

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then we multiply out

$$\begin{aligned} G(x) &= (x^{15} - 2x^{35} + x^{55})\left[\binom{3}{0} + \binom{4}{1}x + \cdots + \binom{k+3}{k}x^k + \cdots\right] \\ &= \left[\binom{3}{0}x^{15} + \binom{4}{1}x^{16} + \cdots + \binom{k+3}{k}x^{k+15} + \cdots\right] \\ &\quad + \left[-2\binom{3}{0}x^{35} - 2\binom{4}{1}x^{36} - \cdots - 2\binom{k+3}{k}x^{k+35} + \cdots\right] \\ &\quad + \left[\binom{3}{0}x^{55} + \binom{4}{1}x^{56} + \cdots + \binom{k+3}{k}x^{k+55} + \cdots\right] \end{aligned}$$

Now x^n occurs at most once in each set of brackets. It occurs where $k = n - 15$ in the first, where $k = n - 35$ in the second and where $k = n - 55$ in the third. Adding these three terms together we get:

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and finally, if $n < 15$ then $c_n = 0$.

A worked-out example

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A minimal, but complete answer is:

$$G(x) = \left(\frac{1}{1-x}\right)\left(\frac{x^5}{1-x}\right)\left(\frac{1-x^{20}}{1-x}\right)\left(\frac{x^{10}(1-x^{20})}{1-x}\right) = \frac{x^{15} - 2x^{35} + x^{55}}{(1-x)^4}$$

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$x^{15}\binom{90+3}{90}x^{90} - 2x^{35}\binom{70+3}{70}x^{70} + x^{55}\binom{50+3}{50}x^{50}$. And so the number of solutions is $\binom{93}{90} - 2\binom{73}{70} + \binom{53}{50}$.

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Answer: $G(x) = (x^{20} - x^{30} - x^{40} + x^{50}) \sum_{j=0}^{\infty} \binom{j+2}{j} x^j$

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Two things to remember: If the condition is $a \leq w_j \leq b$, the part of the generating function coming from w_j is

$$\begin{aligned} x^a + x^{a+1} + \cdots + x^b &= \frac{x^a}{1-x} - \frac{x^{b+1}}{1-x} \\ &= \frac{x^a - x^{b+1}}{1-x}. \end{aligned}$$

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Note that the first expression is the sum of all powers corresponding to allowed values of w_j. In the numerator of the last expression is the first term that appears there, minus the first missing term.

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(Replacing the x in $1 + x + x^2 + x^3 + \cdots = 1/(1 - x)$ with (x^2).)

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(Replacing the x in $1 + x + x^2 + x^3 + \cdots = 1/(1 - x)$ with (x^2).) Similarly, the condition that y_2 is a multiple of 3 has the generating function

$$1 + x^3 + x^6 + x^9 + \cdots = \frac{1}{1 - x^3}.$$

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$$0 \le x_1 \le 14$$
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$$6 \le x_3$$
$$0 \le x_4$$
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$$0 \le x_5$$

Answer: For x_1 we get: $1 + x + x^2 + \cdots + x^{14} = \dfrac{1-x^{15}}{1-x}$.

Still another worked-out example

(a) Find the generating function for the number of solutions of the following equation where the variables x_j are required to be integers satisfying the stated conditions. Your answer should be a single fraction whose denominator is a power of $(1-x)$ and whose numerator is a sum of powers of x.

$$x_1 + x_2 + x_3 + x_4 + x_5 = n$$
$$0 \le x_1 \le 14$$
$$14 \le x_2 \le 28$$
$$6 \le x_3$$
$$0 \le x_4$$
$$0 \le x_5$$

Answer: For x_1 we get: $1 + x + x^2 + \cdots + x^{14} = \dfrac{1-x^{15}}{1-x}$.

For x_2: $x^{14} + x^{15} + \cdots + x^{28} = \dfrac{x^{14}-x^{29}}{1-x}$.

For x_3: $x^6 + x^7 + x^8 + \cdots = \dfrac{x^6}{1-x}$.

For x_3: $x^6 + x^7 + x^8 + \cdots = \dfrac{x^6}{1-x}$.

For x_4 and x_5: $1 + x + x^2 + \cdots = \dfrac{1}{1-x}$.

For x_3: $x^6 + x^7 + x^8 + \cdots = \dfrac{x^6}{1-x}$.

For x_4 and x_5: $1 + x + x^2 + \cdots = \dfrac{1}{1-x}$.

Multiply together:

$$G(x) = \frac{1-x^{15}}{1-x}\,\frac{x^{14}-x^{29}}{1-x}\,\frac{x^6}{1-x}\,\frac{1}{1-x}\,\frac{1}{1-x} = \frac{x^{20}-2x^{35}+x^{50}}{(1-x)^5}$$

(b) Use this generating function to find the number of solutions when $n = 90$.

For x_3: $x^6 + x^7 + x^8 + \cdots = \dfrac{x^6}{1-x}$.

For x_4 and x_5: $1 + x + x^2 + \cdots = \dfrac{1}{1-x}$.

Multiply together:

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(b) Use this generating function to find the number of solutions when $n = 90$.

Answer: $G(x) = (x^{20} - 2x^{35} + x^{50}) \sum_{j=0}^{\infty} \binom{j+4}{j} x^j$.

For x_3: $x^6 + x^7 + x^8 + \cdots = \dfrac{x^6}{1-x}$.

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Answer: $G(x) = (x^{20} - 2x^{35} + x^{50}) \sum_{j=0}^{\infty} \binom{j+4}{j} x^j$. We see that x^{90} shows up in

$$x^{20}\binom{74}{70}x^{70} - 2x^{35}\binom{59}{55}x^{55} + x^{50}\binom{44}{40}x^{40}$$

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$$G(x) = \frac{1-x^{15}}{1-x}\,\frac{x^{14}-x^{29}}{1-x}\,\frac{x^6}{1-x}\,\frac{1}{1-x}\,\frac{1}{1-x} = \frac{x^{20}-2x^{35}+x^{50}}{(1-x)^5}$$

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$$x^{20}\binom{74}{70}x^{70} - 2x^{35}\binom{59}{55}x^{55} + x^{50}\binom{44}{40}x^{40}$$

and so the number of solutions is $\dbinom{74}{70} - 2\dbinom{59}{55} + \dbinom{44}{40}$