Generating Functions

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We will be dealing with functions that are sums of a very large number of powers of x, or even an infinite number. The applications we will put them to are based on the on the rule for multiplying sums of powers:
$(a_0 + a_1x + a_2x^2 + \cdots)(b_0 + b_1x + b_2x^2 + \cdots) = (c_0 + c_1x + c_2x^2 + \cdots)$
where

$$
\begin{aligned}
c_0 &= a_0b_0 \\
c_1 &= a_0b_1 + a_1b_0 \\
c_2 &= a_0b_2 + a_1b_1 + a_2b_0 \\
&\cdots \\
c_n &= a_0b_n + a_1b_{n-1} + a_2b_{n-2} + \cdots + a_nb_0
\end{aligned}
$$

Or, in the $\sum$ notation:

$$\left(\sum_{n=0}^{\infty} a_n x^n\right)\left(\sum_{n=0}^{\infty} b_n x^n\right) = \left(\sum_{n=0}^{\infty} c_n x^n\right)$$

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If the a_j represents the number of ways of doing something with j objects and b_{n-j} represents the number of ways of doing something with the rest of n objects, then $a_j b_{n-j}$ might represent the number of ways of handling all n objects using j first and then $n-j$.

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The utility of generating functions relies on this being true in a large number of situations.

We will focus on one particular type of problem, exemplified by this example: Suppose we have an equation involving variables y_1, y_2, y_3 which are all whole numbers and all greater than or equal to 0:

$$y_1 + y_2 + y_3 = k$$

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Without any conditions, the answer is $c_k = C(k + 3 - 1, k)$, because we can think of this as selecting k times with repetition from a set of size 3. Any solution corresponds to a selection that picks the first element of the set y_1 times, the second y_2 times and the third y_3 times.

One way to attack it is to break it into two equations:

$$y_1 = i \quad \text{with the first condition,}$$

$$y_2 + y_3 = j \quad \text{with the other 2 conditions.}$$

with $i + j = k$.

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Typically we find $F_1(x) = \sum_{i=0}^{\infty} a_i x^i$ and put it in some simple form, then do the same for $F_2(x) = \sum_{j=0}^{\infty} b_j x^j$.

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For example. if there are no conditions imposed, then $y_1 = i$ has one solution for any i. That is $a_i = 1$.
Then $F_1(x) = 1 + x + x^2 + x^3 + \cdots = 1/(1-x)$.
It turns out that $b_j = j + 1$ and $F_2(x) = 1 + 2x + 3x^2 + \cdots = 1/(1-x)^2$.

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$$\frac{1}{(1-x)^3} = \sum_{k=0}^{\infty} \binom{k+2}{k} x^k$$

From which we conclude $c_k = \binom{k+2}{k}$. This is the formula we had previously: $C(k+3-1, k)$.

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We can take this one step further: Divide the equation $y_2 + y_3 = j$ into two equations

$$y_2 = p \quad \text{with the condition on } y_2$$
$$y_3 = q \quad \text{with the condition on } y_3$$

with $p + q = j$. Then, in the case where there are no conditions, a similar analysis will give $1/(1-x)$ for the first equation and $1/(1-x)$ for the second equation, leading to $F_2(x) = 1/(1-x)^2$.

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Do the same for the second, third, etc., parts getting $F_2(x)$, $F_3(x)$, etc.
For the right kind of problem the numbers associated with the whole problem will have the generating function $G(x) = F_1(x)F_2(x)F_3(x)\cdots$.

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Finally, analyse $G(x)$ and discover the desired numbers.

For the process to work, we need a repertoire of formulas for various sequences and their generating functions. Here are some of the most important from the textbook, page 424.

$$(1) \qquad (1+x)^n = \binom{n}{0} + \binom{n}{1}x + \binom{n}{2}x^2 + \cdots + \binom{n}{n}x^n = \sum_{j=0}^{n} \binom{n}{j} x^j$$

$$(2) \qquad \frac{1-x^{n+1}}{1-x} = 1 + x + x^2 + \cdots + x^n = \sum_{j=0}^{n} x^j$$

$$(3) \qquad \frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots = \sum_{j=0}^{\infty} x^j$$

$$(4) \qquad \frac{1}{(1-x)^m} = 1 + mx + \binom{m+1}{2}x^2 + \binom{m+2}{3}x^3 + \cdots = \sum_{j=0}^{\infty} \binom{j+m-1}{j} x^j$$

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The rest of the formulas in the textbook can be obtained by substitution. For example, replacing x by (ax) in (3) gives

$$(5) \qquad \frac{1}{1-ax} = 1 + ax + (ax)^2 + (ax)^3 + \cdots = \sum_{j=1}^{\infty} a^j x^j$$

The equation (1) tells us that the generating function of the sequence $\binom{n}{0}, \binom{n}{1}, \binom{n}{2}, \ldots, \binom{n}{n}, 0, 0, \ldots$ is $(1+x)^n$.

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The equation (3) tells us that the generating function of the sequence $1, 1, 1, 1, \ldots$ is $1/(1-x)$. And equation (5) tells us that the generating function of the sequence $1, a, a^2, a^3, \ldots$ is $1/(1-ax)$.

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While these formulas may seem to have come out of thin air, they can actually be derived in relatively simple ways.

The equation (1) tells us that the generating function of the sequence $\binom{n}{0}, \binom{n}{1}, \binom{n}{2}, \ldots, \binom{n}{n}, 0, 0, \ldots$ is $(1+x)^n$.
The equation (3) tells us that the generating function of the sequence $1, 1, 1, 1, \ldots$ is $1/(1-x)$. And equation (5) tells us that the generating function of the sequence $1, a, a^2, a^3, \ldots$ is $1/(1-ax)$.

While these formulas may seem to have come out of thin air, they can actually be derived in relatively simple ways.

The first equation is just the *binomial theorem*:

$$(x+y)^n = \sum_{j=0}^{n} \binom{n}{j} x^j y^{n-j}$$

with y set equal to 1.

To see where (2) comes from:

$$
\begin{aligned}
\text{Let} \qquad && G(x) &= 1 + x + x^2 + x^3 + \cdots + x^n \\
\text{multiply} \qquad && xG(x) &= \phantom{1 + {}} x + x^2 + x^3 + \cdots + x^n + x^{n+1} \\
\text{subtract} \qquad && (1-x)G(x) &= 1 - x^{n+1} \\
\text{divide} \qquad && G(x) &= \frac{1 - x^{n+1}}{1 - x}
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Equation (3) comes about the same way:

$$
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\text{Let} \qquad & F(x) = 1 + x + x^2 + x^3 + \cdots \\
\text{multiply} \qquad & xF(x) = \qquad x + x^2 + x^3 + \cdots \\
\text{subtract} \qquad & (1-x)F(x) = 1 \\
\text{divide} \qquad & F(x) = \frac{1}{1 - x}
\end{aligned}
$$