# More Rook Polynomials

Daniel H. Luecking

February 7, 2024

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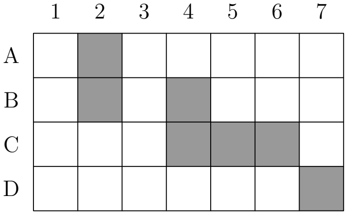

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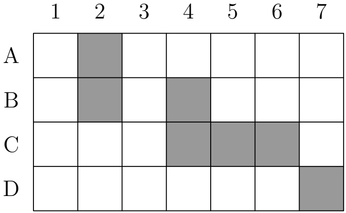


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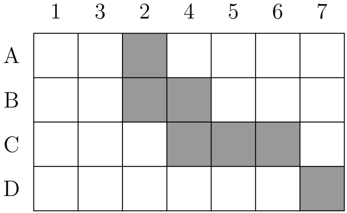

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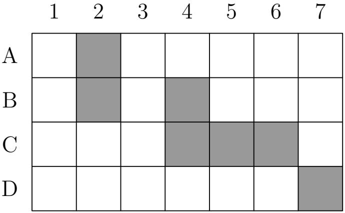


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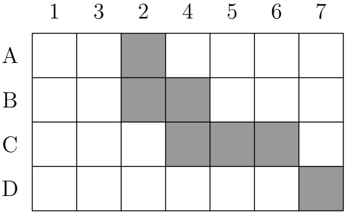


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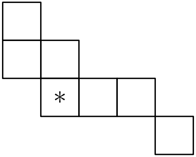
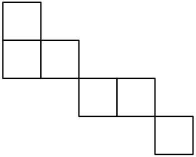
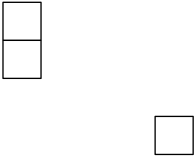
$C =$ 


$C_e =$ 

$C_s =$ 

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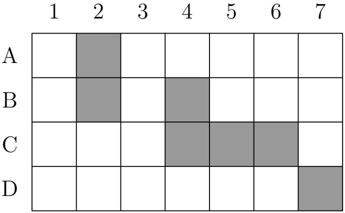


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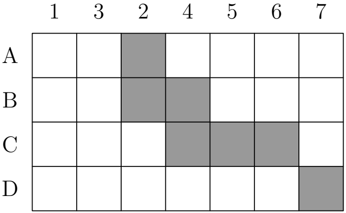


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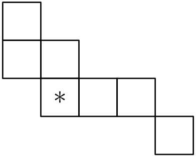


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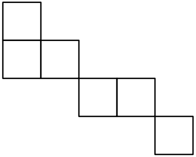

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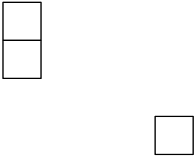

We can use the product formula on $C_e$ and $C_s$ to get

$$r(C_e, x) = (1 + 3x + x^2)(1 + 3x + 2x^2) = 1 + 6x + 12x^2 + 9x^3 + 2x^4$$

$$r(C_s, x) = (1 + 2x)(1 + x) = 1 + 3x + 2x^2$$

So the formula gives us

$$r(C,x) = r(C_e,x) + x \cdot r(C_s,x) = 1 + 7x + 15x^2 + 11x^3 + 2x^4$$

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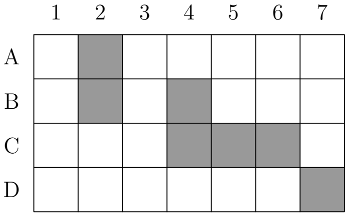
The inclusion-exclusion method requires us to find $N = P(7, 4)$, $S_1 = 7P(6, 3)$, $S_2 = 15P(5, 2)$, $S_3 = 11P(4, 1)$ and $S_4 = 2P(3, 0)$



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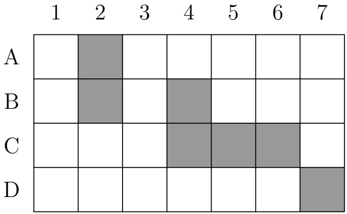


Then

$$\begin{aligned} N(\overline{c_1}\,\overline{c_2}\,\overline{c_3}\,\overline{c_4}) &= N - S_1 + S_2 - S_3 + S_4 \\ &= P(7,4) - 7P(6,3) + 15P(5,2) - 11P(4,1) + 2P(3,0) \\ &= \frac{7!}{3!} - 7\,\frac{6!}{3!} + 15\,\frac{5!}{3!} - 11\,\frac{4!}{3!} + 2\,\frac{3!}{3!} = 258 \end{aligned}$$

## Examples of finding rook polynoials

This chessboard is from page 405 of the textbook, but I've chosen a different square to mark.

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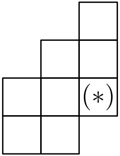

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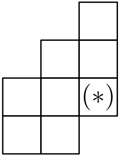
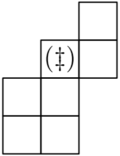
$C$ = 

with $C_e$ = 

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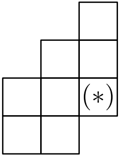
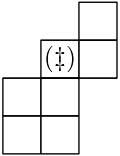
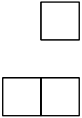
$C =$ 

with $C_e =$ 

and $C_s =$ 

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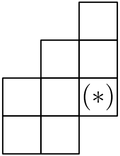
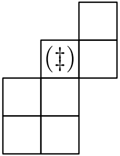
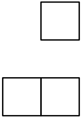
$C =$ 

with $C_e =$ 

and $C_s =$ 

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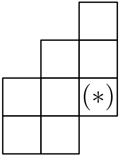
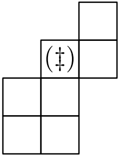
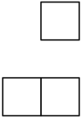
$C$ = 

with $C_e$ = 

and $C_s$ = 

$C_e$ also has a square marked because we'll be using the square-removal formula for $C_e$ as well.

Thus, $r(C_e, x) = (1 + 4x + 2x^2)(1 + 2x) + x(1 + 2x)(1 + x)$
and $r(C_s, x) = 1 + 3x + x^2$.

Finally, $r(C, x) = r(C_e, x) + x \cdot r(C_s, x) = 1 + 8x + 16x^2 + 7x^3$.

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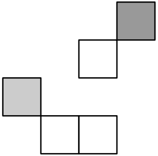
$C =$ 

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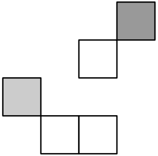
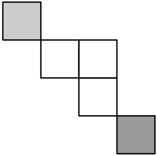
$C =$  $\longrightarrow$ $C' =$ 

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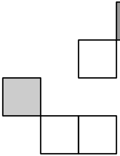
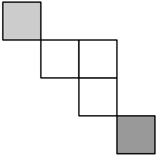
$C =$  $\longrightarrow$ $C' =$ 

Then we can apply the product formula twice to get

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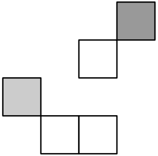
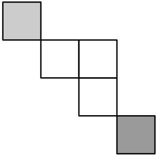
$C =$  $\longrightarrow$ $C' =$ 

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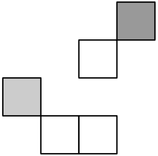
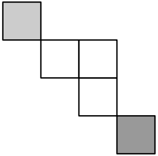
$C =$  $\longrightarrow \ C' =$ 

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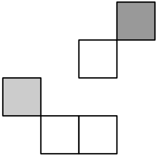
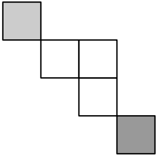
$C =$  $\longrightarrow$ $C' =$ 

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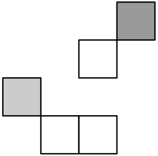
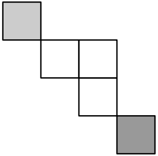
$C =$  $\longrightarrow$ $C' =$ 

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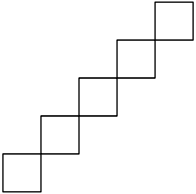
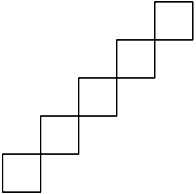
Final example: $C =$  .

Final example: $C =$  .

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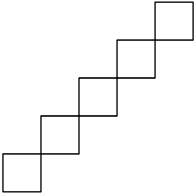
Final example: $C =$  .

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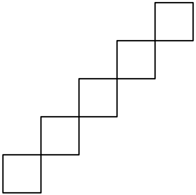
Final example: $C =$  .

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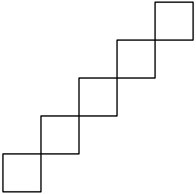
Final example: $C =$  .

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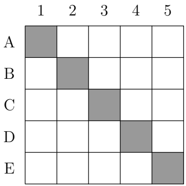
Either way we get

$$r(C, x) = \binom{5}{0} + \binom{5}{1}x + \binom{5}{2}x^2 + \binom{5}{3}x^3 + \binom{5}{4}x^4 + \binom{5}{5}x^5$$
$$= 1 + 5x + 10x^2 + 10x^3 + 5x^4 + x^5.$$

## Derangements revisited




Suppose 5 people initially sat in the seats indicated by the shaded squares, and then they all decided they wanted a different seat.

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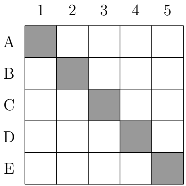


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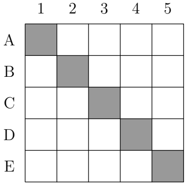


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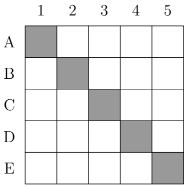


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The rook polynomial of the shaded chessboard is

$$(1+x)^5 = 1 + 5x + \binom{5}{2}x^2 + \binom{5}{3}x^3 + \binom{5}{4}x^4 + \binom{5}{5}x^5$$

The formula for the number of ways to seat the people is

$$P(5,5) - 5P(4,4) + \binom{5}{2}P(3,3) - \binom{5}{3}P(2,2) + \binom{5}{4}P(1,1) - \binom{5}{5}P(0,0)$$

Filling in the numbers:

$$5! - 5 \cdot 4! + \frac{5!}{2!} - \frac{5!}{3!} + \frac{5!}{4!} - \frac{5!}{5!}$$