Forbidden Positions

Daniel H. Luecking

February 5, 2024

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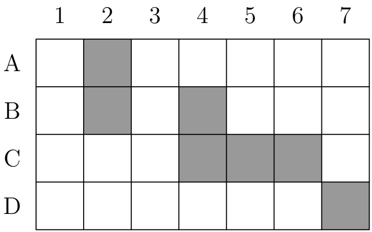

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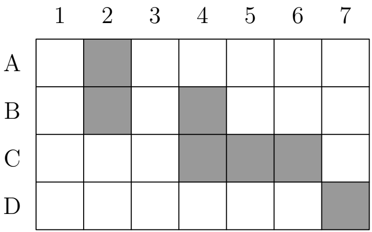

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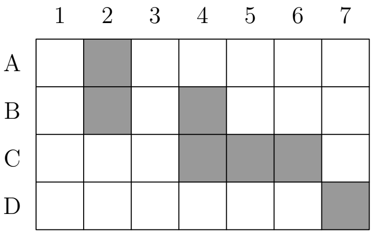

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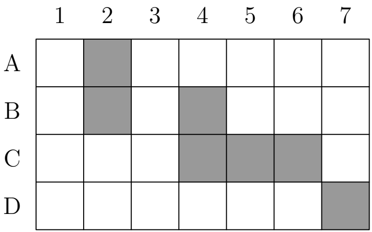

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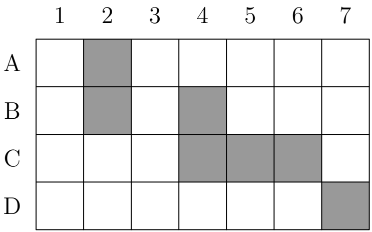

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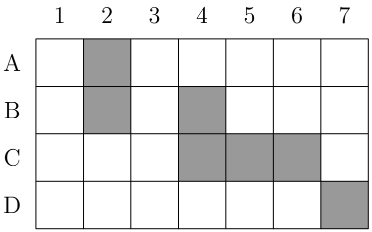

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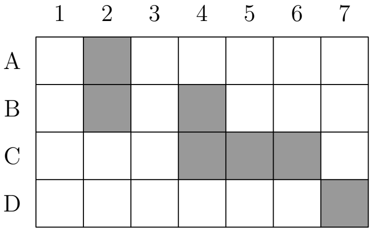
So, $S_1 = 1 \cdot P(6, 3) + 2 \cdot P(6, 3) + 3 \cdot P(6, 3) + 1 \cdot P(6, 3)$ or $S_1 = 7 \cdot P(6, 3)$.

Using the rule of product again, we can get $N(c_1c_2) = 1 \cdot P(5,2)$. The number 1 is the number of ways to seat A and B in forbidden seats and $P(5,2)$ is the number of ways to seat the other 2 in the remaining 5 seats.



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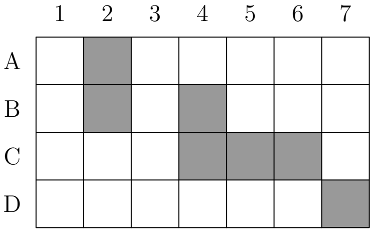

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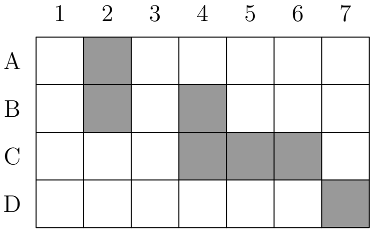

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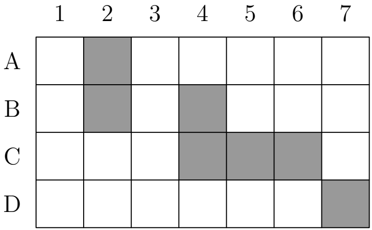

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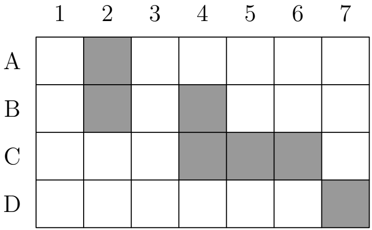

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and $S_4 = 2P(3,0)$, where

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Note that if $k = 8$ this equals $1 \cdot 8!$. And if $k = 0$ this equals $1 \cdot 1$.

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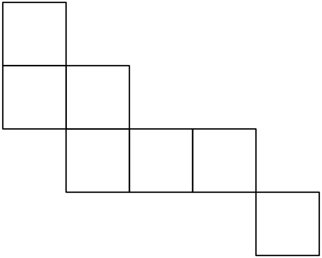
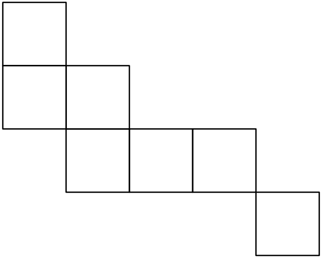


$C =$ 

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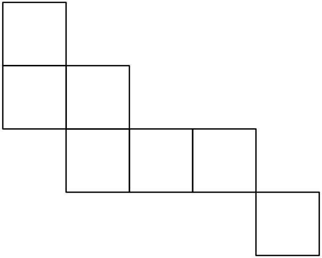
$C =$ 

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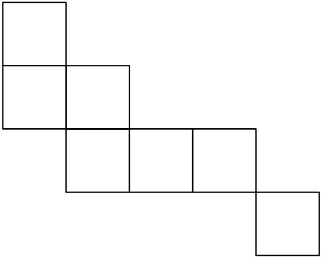
Thus we have seen that $r_1(C) = 7$, $r_2(C) = 15$, $r_3(C) = 11$, and $r_4(C) = 2$. To be complete $r_0(C) = 1$ and $r_k(C) = 0$ for all $k \geq 5$.

$C =$ 

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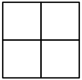
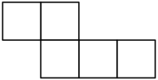
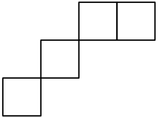
Some examples:

$C_1 =$  $C_2 =$  $C_3 =$ 

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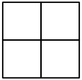
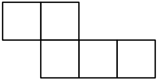
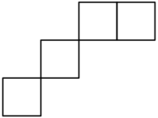
$C_1 =$  $C_2 =$  $C_3 =$ 

Then

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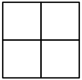
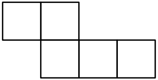
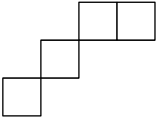
$C_1 =$  $C_2 =$  $C_3 =$ 

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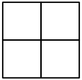
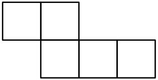
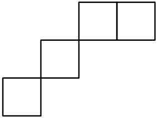
$C_1 =$  $C_2 =$  $C_3 =$ 

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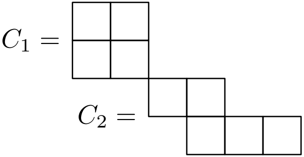
$$r(C, x) = r_0 + r_1 x + r_2 x^2 + \cdots$$

Thus, each power x^k is multiplied by $r_k(C) = r_k$. Since there are only finitely many terms, this is always a polynomial. For example

$$r(C_3, x) = 1 + 4x + 5x^2 + 2x^3$$

Consider this chessboard: $C =$



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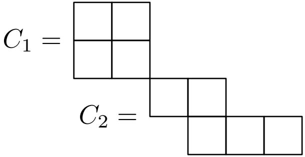

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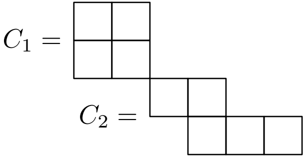

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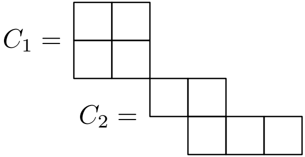

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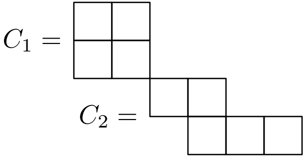

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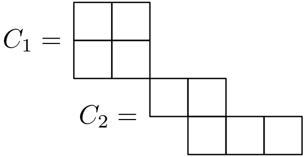

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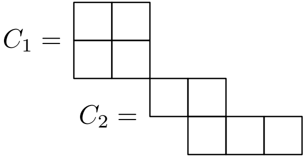

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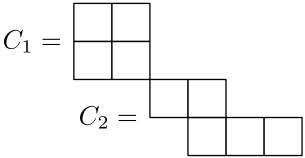

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This is the formula for the x^2 term in $r(C_1, x)r(C_2, x)$. In fact,

$$\begin{aligned} r(C, x) = r(C_1, x)r(C_2, x) &= (1 + 4x + 2x^2)(1 + 5x + 5x^2) \\ &= 1 + 9x + 27x^2 + 30x^3 + 10x^4 \end{aligned}$$

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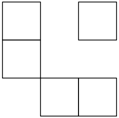

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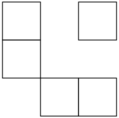

This is useful by itself, when applicable, but we need another tool for computing $r(C, x)$ that is applicable even when this one is not.

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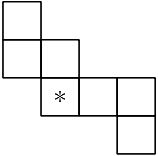
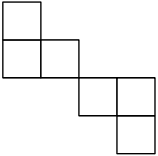
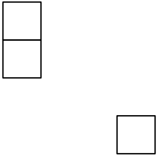
$C =$ 

$C_e =$ 
$C_s =$ 

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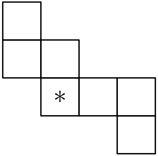
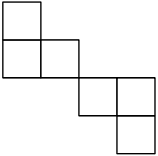
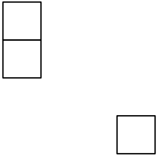
C =  C_e =  C_s = 

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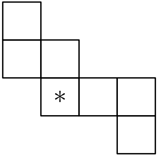
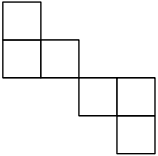
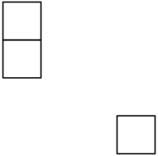
$C =$ 

$C_e =$ 
$C_s =$ 

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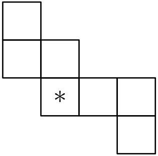
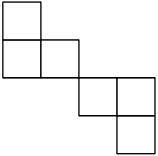
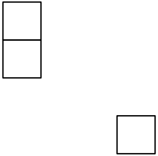
$C =$ 

$C_e =$ 
$C_s =$ 

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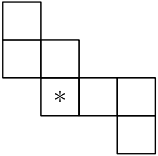
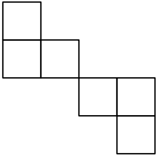
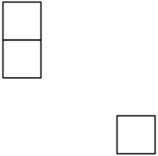
$C =$ 

$C_e =$ 
$C_s =$ 

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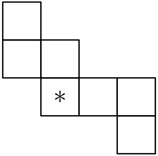
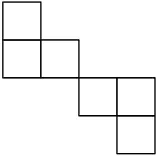
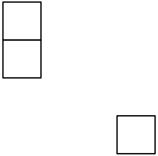
C = 

C_e = 
C_s = 

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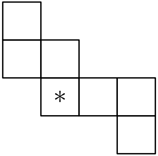
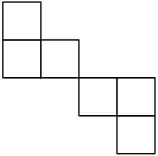
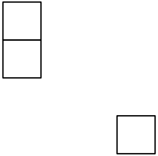
$C =$ 

$C_e =$ 
$C_s =$ 

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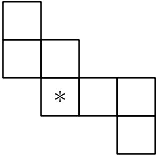
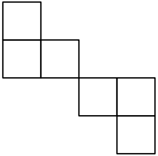
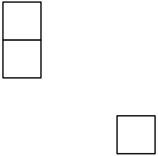
$C =$ 

$C_e =$ 
$C_s =$ 

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Adding these we get the formula for $r(C, x)$:

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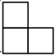
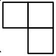
$r(C_e, x) = r\Big($  $, x\Big)\, r\Big($  $, x\Big) = (1 + 3x + x^2)^2$

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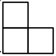
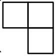
$r(C_e, x) = r\Big($  $, x\Big)\, r\Big($  $, x\Big) = (1 + 3x + x^2)^2$

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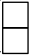
$r(C_s, x) = r\Big($  $, x\Big)\, r\big($  $, x\big) = (1 + 2x)(1 + x)$

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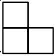
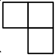
$r(C_e, x) = r\,($  $, x)\; r\,($  $, x) = (1 + 3x + x^2)^2$

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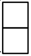
$r(C_s, x) = r\,($  $, x)\; r\,($  $, x) = (1 + 2x)(1 + x)$

So that,

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r(C, x) &= r(C_e, x) + x \cdot r(C_s, x) \\
r(C, x) &= (1 + 3x + x^2)^2 + x(1 + 2x)(1 + x) \\
&= 1 + 7x + 14x^2 + 8x^3 + x^4
\end{aligned}
$$