Derangements

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In particular, we get the approximation $d_n \approx n!/e$ and the probability of a derangement is approximately $1/e$. This is accurate to at least 6 decimal places for $n \geq 10$.

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Despite the previous discussion, producing the value $10!\,/e$ for d_{10} will not be accepted as correct, as it is only approximate.

In the analysis of derangements via inclusion-exclusion, when we have n objects being permuted, we have the formula $S_k = \dfrac{n!}{k!}$.

This simple formula for S_k allows us to quickly obtain the number of permutations that have at least 3 objects in their original place:

$$\begin{aligned} L_3 &= S_3 - \binom{3}{1} S_4 + \binom{4}{2} S_5 - \cdots \pm \binom{n-1}{n-3} S_n \\ &= \frac{n!}{3!} - 3\,\frac{n!}{4!} + 6\,\frac{n!}{5!} - \cdots \pm \frac{(n-1)!}{2!\,(n-3)!}\,\frac{n!}{n!} \end{aligned}$$

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However there is an easier way to compute E_k. Since E_k is the number of permutations with exactly k objects in their original positions, we can create such permutations in two steps:

1. Pick which k positions to leave the same: $C(n, k)$ ways.
2. Perform a derangement of the remaining $n - k$ objects: d_{n-k} ways.

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By the rule of product, $E_k = C(n,k) d_{n-k} = \dfrac{n!}{k!\,(n-k)!} d_{n-k}$. Compare that to the formula

$$E_k = \frac{n!}{k!} - \binom{k+1}{1}\frac{n!}{(k+1)!} + \binom{k+2}{2}\frac{n!}{(k+2)!} - \cdots \pm \binom{n}{n-k}\frac{n!}{n!}.$$

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How many ways can they all get the wrong phone? This is a derangement of the original function associating owner to phone. There are d_7 of those and

$$d_7 = 7! \left(\frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} - \frac{1}{5!} + \frac{1}{6!} - \frac{1}{7!} \right)$$

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This is every permutation except the derangements and there are $7! - d_7$ of these.

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This is every permutation except the derangements and there are $7! - d_7$ of these.

How many ways can exactly 3 students get their own phone?
This is $E_3 = C(7,3)d_4 = \dfrac{7!}{3!\,4!}\,4! \left(\dfrac{1}{2!} - \dfrac{1}{3!} + \dfrac{1}{4!} \right)$.

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This is a derangement, so the number of ways is d_6. One way to write this is

$$\begin{aligned} d_6 &= \frac{6!}{2!} - \frac{6!}{3!} + \frac{6!}{4!} - \frac{6!}{5!} + \frac{6!}{6!} \\ &= 6 \cdot 5 \cdot 4 \cdot 3 - 6 \cdot 5 \cdot 4 + 6 \cdot 5 - 6 + 1 \\ &= 6(5(4(3-1)+1)-1)+1 \end{aligned}$$

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I wrote the last line to illustrate the most efficient way to calculate d_6. In general, if you need to calculate d_n efficiently, a recursive method based on $d_n = nd_{n-1} + (-1)^n$ is probably best. For example $d_2 = 1$, so $d_3 = 3 \cdot d_2 - 1 = 2$, $d_4 = 4 \cdot d_3 + 1 = 9$, $d_5 = 5 \cdot d_4 - 1 = 44$, and so on.

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How many ways could at most half of them be wrong? This means at least 3 are correct so

$$L_3 = \frac{6!}{3!} - \binom{3}{1}\frac{6!}{4!} + \binom{4}{2}\frac{6!}{5!} - \binom{5}{3}\frac{6!}{6!} = 120 - 90 + 36 - 10 = 56.$$

Selections with Forbidden Choices

It is not hard to determine how many ways 4 people can be placed in 7 seats: Pick 4 seats and then assign each to one of the persons. This is a permutation and so there are $P(7,4) = 7!\,/(7-4)!$ ways.

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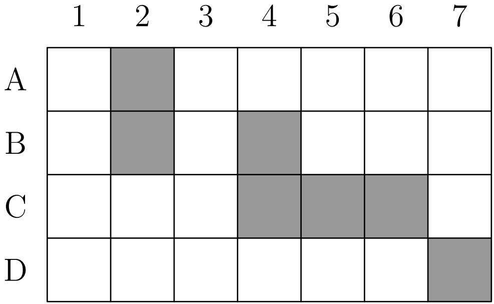
But suppose some persons refuse to sit in some seats? Here is a visual representation where a shaded square indicates that the person on the left refuses to sit in the seat above.



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What we want is $N(\overline{c_1}\,\overline{c_2}\,\overline{c_3}\,\overline{c_4}) = N - S_1 + S_2 - S_3 + S_4$. It turns out that finding N and S_1 are not hard, finding other S_k takes some work.