Extensions of Inclusion-Exclusion

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Inclusion-Exclusion

For m conditions $c_1, c_2, \ldots, c_m$, we let N be the number of object to which these conditions apply, and we define S_1 through S_m by

1. $S_1 = N(c_1) + N(c_2) + N(c_3) + \cdots + N(c_m)$.
2. $S_2 = N(c_1c_2) + N(c_1c_3) + N(c_2c_3) + \cdots + N(c_{m-1}c_m)$.
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In some cases all the numbers in a sum are the same, so it is useful to know that S_1 has m terms, S_2 has $C(m, 2)$ terms. In general any S_k has $C(m, k)$ terms.

The formula for the number that satisfy at least one condition is

$$S_1 - S_2 + S_3 - \cdots \pm S_m$$

and the number that satisfy none of the conditions is

$$N - S_1 + S_2 - S_3 + \cdots \mp S_m$$

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Another easy one is $L_0 = N$. We also have $E_m = L_m = S_m = N(c_1 c_2 \ldots c_m)$ and if $k < m$, $E_k = L_k - L_{k+1}$.

To illustrate some of the issues involved, if there are 3 conditions we have

$$E_1 = N(c_1\overline{c_2}\,\overline{c_3}) + N(\overline{c_1}c_2\overline{c_3}) + N(\overline{c_1}\,\overline{c_2}c_3)$$

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This extends to any number of conditions

$$E_1 = S_1 - 2S_2 + 3S_3 - 4S_4 + \cdots \pm mS_m$$

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If there are m conditions and $0 \leq k \leq m$ then

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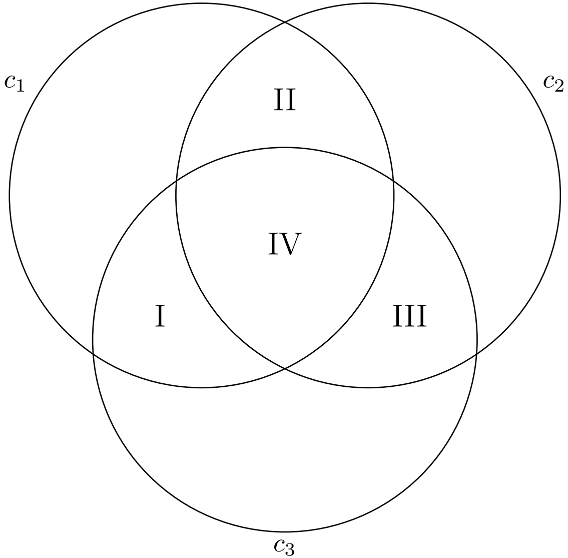

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(a) $E_1 = S_1 - 2S_2 + 3S_3 = 3 \cdot 8! - 2 \cdot 3 \cdot 7! + 3 \cdot 6! = 92{,}880.$

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(c) $L_2 = S_2 - 2S_3 = 3 \cdot 7! - 2 \cdot 6! = 13{,}680.$

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Returning to the arrangements of "BOOKBINDING" with conditions $c_1 =$ ‘contains "BB"’, $c_2 =$ ‘contains "OO"’, $c_3 =$ ‘contains "II"’, and $c_4 =$ ‘contains "NN"’.

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(a) How many contain exactly 2 of those substrings? Looking up

$$E_2 = S_2 - \binom{3}{1}S_3 + \binom{4}{2}S_4 = S_2 - 3S_3 + 6S_4 = 6\,\frac{9!}{2!\,2!} - 3\cdot 4\,\frac{8!}{2!} + 6\cdot 7!$$

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(c) At least 2: $L_2 = S_2 - \binom{2}{1}S_3 + \binom{3}{2}S_4 = 6\,\frac{9!}{2!\,2!} - 2\cdot 4\,\frac{8!}{2!} + 3\cdot 7!$

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$10\,\frac{10!}{2!\,2!\,2!} - 3\cdot 10\,\frac{9!}{2!\,2!} + 6\cdot 5\frac{8!}{2!} - 10\cdot 7!$.

Arrangements of the string "VETERINARIAN" with 5 conditions about containing substrings "EE", "RR", "II", "NN", "AA".

We found previously that $S_1 = 5\frac{11!}{2!\,2!\,2!\,2!}$, $S_2 = 10\frac{10!}{2!\,2!\,2!}$, $S_3 = 10\frac{9!}{2!\,2!}$, $S_4 = 5\frac{8!}{2!}$, and $S_5 = 7!$.

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(c) At least 3?

$$L_3 = S_3 - \binom{3}{1}S_4 + \binom{4}{2}S_5 = S_3 - 3S_4 + 6S_5 = 10\,\frac{9!}{2!\,2!} - 3\cdot 5\frac{8!}{2!} + 6\cdot 7!$$

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If we have a given permutation of all the elements of a set, then the permutations that are different from it in every position are called *derangements* of that permutation.

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If A is the set of men and B is the set of hats we have the original function assigning to each man his own hat. Afterwards we have the new function that assigns to each man the hat he is handed. To compute the probability we need to divide the number of derangements by the number of all one-to-one functions.

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By a similar argument, for any two conditions $N(c_j c_k) = 13!$. If we take the sum of all these (there are $C(15, 2)$ terms) we get
$S_2 = \binom{15}{2} 13! = \frac{15!}{2!\,13!} 13! = \frac{15!}{2!}$.

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$$\begin{aligned} N(\overline{c_1}\,\overline{c_2}\ldots\overline{c_{15}}) &= N - S_1 + S_2 - S_3 + S_4 - \cdots - S_{15} \\ &= 15! - 15! + \frac{15!}{2!} - \frac{15!}{3!} + \frac{15!}{4!} - \cdots - \frac{15!}{15!} \\ &= 15!\left(\frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} - \cdots - \frac{1}{15!}\right) \end{aligned}$$

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So the probability of a derangement is $\dfrac{1}{2!} - \dfrac{1}{3!} + \cdots - \dfrac{1}{15!}$ which is approximately 0.36788.

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So the probability of a derangement is $\dfrac{1}{2!} - \dfrac{1}{3!} + \cdots - \dfrac{1}{15!}$ which is approximately 0.36788.

The general formula for the number of derangements of a permutation with length n is

$$d_n = \frac{n!}{2!} - \frac{n!}{3!} + \frac{n!}{4!} - \cdots \pm \frac{n!}{n!}$$