Inclusion-Exclusion

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The Principle of Inclusion and exclusion

If we carefully examine what gets counted and how often, we can extend the formula $|A \cup B| = |A| + |B| - |A \cap B|$ to three sets as follows

$$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|.$$

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Another way to establish this is to apply the formula for a pair of sets to the pair A and $B \cup C$ to get (as a first step)

$$|A \cup (B \cup C)| = |A| + |B \cup C| - |A \cap (B \cup C)|$$

Then apply the formula again to $|B \cup C| = |B| + |C| - |B \cap C|$ and then again to

$$|A \cap (B \cup C)| = |(A \cap B) \cup (A \cap C)| = |(A \cap B)| + |(A \cap C)| - |A \cap B \cap A \cap C|.$$

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It is tedious, but this approach can lead to a formula (and a proof of it) that holds for any number of sets. Unfortunately, the notation can quickly get out of hand so that even writing down the formula is difficult. Therefore we start with an alternative way to present such problems.

Why we need an alternative approach:

$$\begin{aligned}|A \cup B \cup C \cup D| = {} & |A| + |B| + |C| + |D| \\ & - |A \cap B| - |A \cap C| - |A \cap D| - |B \cap C| - |B \cap D| - |C \cap D| \\ & + |A \cap B \cap C| + |A \cap B \cap D| + |A \cap C \cap D| + |B \cap C \cap D| \\ & - |A \cap B \cap C \cap D|\end{aligned}$$

We start with a collection of objects. We don't give that collection a name, but we use N for the number of elements it contains. We assume there are subsets of this collection defined by conditions. Lets walk through an example to illustrate what this means.

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Lets consider all math majors at some college. Lets say there are $N = 114$ of them. Suppose, for example, we have conditions like the following:

1. c_1 is the condition that a student is enrolled in Algebra.
2. c_2 is the condition that a student is enrolled in Geometry.
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3. c_3 is the condition that a student is enrolled in Calculus.

Then the students that satisfy c_1 are the subset of students taking Algebra, call this subset A. Let B be the subset of those taking Geometry (condition c_2) and let C be the subset of those taking Calculus (condition c_3).

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Then the students that satisfy c_1 are the subset of students taking Algebra, call this subset A. Let B be the subset of those taking Geometry (condition c_2) and let C be the subset of those taking Calculus (condition c_3). Moreover, the set of students that satisfy at least one of these conditions is the set of student that are enrolled in at least one of these courses and that is $A \cup B \cup C$.

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Then the students that satisfy c_1 are the subset of students taking Algebra, call this subset A. Let B be the subset of those taking Geometry (condition c_2) and let C be the subset of those taking Calculus (condition c_3). Moreover, the set of students that satisfy at least one of these conditions is the set of student that are enrolled in at least one of these courses and that is $A \cup B \cup C$. Also, for example, the set of students that satisfy both c_1 and c_2 is the intersection of the corresponding sets: $A \cap B$.

We use $N(c_1)$ to stand for the number of students that satisfy c_1, so $N(c_1) = |A|$. $N(c_2)$ for the number of students that satisfy c_2, and so on.

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Finally, we use $\overline{c_j}$ for the opposite of c_j and, for example $N(c_1\overline{c_4}\overline{c_5})$ is the number that satisfy condition c_1 but not c_4 and not c_5.

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The number of math majors taking at least one of these three classes is

$$\begin{aligned} |A \cup B \cup C| &= N(c_1) + N(c_2) + N(c_3) \\ &\quad - N(c_1c_2) - N(c_1c_3) - N(c_2c_3) + N(c_1c_2c_3) \\ &= 30 + 25 + 50 - 8 - 15 - 12 + 5 = 75 \end{aligned}$$

For the set of math majors, suppose $N(c_1) = 30$ (i.e., 30 math majors are enrolled in Algebra), $N(c_2) = 25$, $N(c_3) = 50$. Some students might be enrolled in more than one of these classes so suppose $N(c_1c_2) = 8$, $N(c_1c_3) = 15$, $N(c_2c_3) = 12$ and $N(c_1c_2c_3) = 5$.

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When the number of conditions gets large, these sums and differences get pretty long. We will use the symbol S_1 to stand for $N(c_1) + N(c_2) + N(c_3) = 105$ and S_2 will stand for $N(c_1c_2) + N(c_1c_3) + N(c_2c_3) = 35$ and $S_3 = N(c_1c_2c_3) = 5$.

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The number of math majors taking at least one of these three classes is

$$\begin{aligned}|A \cup B \cup C| &= N(c_1) + N(c_2) + N(c_3) \\ &\quad - N(c_1c_2) - N(c_1c_3) - N(c_2c_3) + N(c_1c_2c_3) \\ &= 30 + 25 + 50 - 8 - 15 - 12 + 5 = 75\end{aligned}$$

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So, the number above is $S_1 - S_2 + S_3 = 105 - 35 + 5 = 75$

In general, if we have m conditions we define S_1, S_2, up to S_m, as follows

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3. $S_3 = N(c_1c_2c_3) + N(c_1c_2c_4) + \cdots + N(c_{m-2}c_{m-1}c_m)$. This is the sum of all $N(c_ic_jc_k)$ for all combinations of 3 conditions. There are $C(m, 3)$ terms in this sum

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   ...

m $S_m = N(c_1c_2 \ldots c_m)$. This is the only combination of all m conditions.

The Inclusion-Exclusion formula

Then our formula for the number that satisfy at least one condition is

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and the number that satisfy none of the conditions is

$$N - S_1 + S_2 - S_3 + \cdots \mp S_m$$

A completely worked out example

Problem: Consider all permutations of the letters of the alphabet. (a) How many contain at least one of the substrings `"COW"`, `"HEN"` or `"PIG"`? (b) How many contain none of these substrings?

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Solution: The number of permutations is $26!$. This is the N of the second formula. We have 3 conditions

- c_1: ‘"COW" is a substring’
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Then $N(c_1)$ is the number of permutations that contain the substring `"COW"`. We've seen that this is $24!$. The same is true for $N(c_2)$ and $N(c_3)$.

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Then $N(c_1)$ is the number of permutations that contain the substring "COW". We've seen that this is $24!$. The same is true for $N(c_2)$ and $N(c_3)$.

Also, $N(c_1c_2)$ is the number of permutations that contain both the substrings "COW" and "HEN" so $N(c_1c_2) = 22!$. The same is true for $N(c_1c_3)$ and $N(c_2c_3)$.

Finally, $N(c_1c_2c_3) = 20!$ and we have

- $S_1 = N(c_1) + N(c_2) + N(c_3) = 3 \cdot 24!\,.$
- $S_2 = N(c_1c_2) + N(c_1c_3) + N(c_2c_3) = 3 \cdot 22!\,.$
- $S_3 = N(c_1c_2c_3) = 20!\,.$

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- $S_3 = N(c_1c_2c_3) = 20!$.

and so the answer to (a), the number of strings that contain at least one of these substrings, is

$$S_1 - S_2 + S_3 = 3 \cdot 24! - 3 \cdot 22! + 20!$$

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$$S_1 - S_2 + S_3 = 3 \cdot 24! - 3 \cdot 22! + 20!$$

and the answer to (b), the number of strings that contain none of these substrings, is

$$N - S_1 + S_2 - S_3 = 26! - 3 \cdot 24! + 3 \cdot 22! - 20!$$

Finally, $N(c_1c_2c_3) = 20!$ and we have

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(Please do not ever give me the answer
401,433,485,490,687,241,912,320,000.)

Another worked out example

Problem: Consider the string `"BOOKBINDING"`, which has length 11 and has a '`B`' in 2 places, an '`O`' in 2 places, an '`I`' in 2 places and an '`N`' in 2 places. How many arrangements of this string have no consecutive duplicate letters?

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Solution: Containing no consecutive 'B's (for example) is the same as not containing the substring "BB". So we are looking for the number of arrangements of "BOOKBINDING" that *do not* satisfy any of the following 4 conditions

- c_1: 'contains the substring "BB"'
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The number of all arrangements is $N = \dfrac{11!}{2!\,2!\,2!\,2!}$

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We can compute $N(c_1)$ by gluing the two 'B's together and ask for the number of arrangements of `"BB"` plus the remaining 9 letters (which contain 3 duplicate pairs).

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The gluing process also produces these numbers

$$\begin{aligned} S_2 &= N(c_1c_2) + N(c_1c_3) + N(c_1c_4) + N(c_2c_3) + N(c_2c_4) + N(c_3c_4) \\ &= 6\,\frac{9!}{2!\,2!} \end{aligned}$$

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Similarly,

$$S_3 = N(c_1c_2c_3) + N(c_1c_2c_4) + N(c_1c_3c_4) + N(c_2c_3c_4) = 4\,\frac{8!}{2!}$$

And the last one is

$$S_4 = N(c_1 c_2 c_3 c_4) = 7!\ .$$

Finally we can get the number of arrangements that satisfy none of the conditions:

$$\begin{aligned} N(\overline{c_1}\,\overline{c_2}\,\overline{c_3}\,\overline{c_4}) &= N - S_1 + S_2 - S_3 + S_4 \\ &= \frac{11!}{2!\,2!\,2!\,2!} - 4\,\frac{10!}{2!\,2!\,2!} + 6\,\frac{9!}{2!\,2!} - 4\,\frac{8!}{2!} + 7! \end{aligned}$$

(This seems to be equal to 1,144,395. But don't ever give me that answer.)

Still another example, with less detail

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Problem: How many permutations of the string "MODERNIST" contain none of the substrings "DO", "RE" or "MI"?

Solution: $c_1 =$'contains "DO"', $c_2 =$'contains "RE"' and $c_3 =$'contains "MI"'.

$N = 9!\,$, $S_1 = 3N(c_1) = 3 \cdot 8!\,$, $S_2 = 3N(c_1c_2) = 3 \cdot 7!$ and $S_3 = N(c_1c_2c_3) = 6!\,$.

Still another example, with less detail

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$N(\overline{c_1}\,\overline{c_2}\,\overline{c_3}) = 9! - 3 \cdot 8! + 3 \cdot 7! - 6!$.

A somewhat less symmetrical example

The string `"HYDROMAGNETICS"` has length 14 and so $14!$ possible arrangements. How many do not contain any of the substrings `"DRONE"`, `"NET"` or `"HYMN"`?

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A large example

How many of the arrangements of the string "VETERINARIAN" contain none of the substrings "EE", "RR", "II", "NN", "AA"?

A somewhat less symmetrical example

The string "HYDROMAGNETICS" has length 14 and so $14!$ possible arrangements. How many do not contain any of the substrings "DRONE", "NET" or "HYMN"?

- $S_1 = 10! + 12! + 11!$
- $S_2 = 9! + 0 + 9!$
- $S_3 = 0$

Answer: $N - S_1 + S_2 - S_3 = 14! - (10! + 12! + 11!) + (9! + 9!) - 0.$

A large example

How many of the arrangements of the string "VETERINARIAN" contain none of the substrings "EE", "RR", "II", "NN", "AA"?

Answer:

$$\frac{12!}{2!\,2!\,2!\,2!\,2!} - \binom{5}{1}\frac{11!}{2!\,2!\,2!\,2!} + \binom{5}{2}\frac{10!}{2!\,2!\,2!} - \binom{5}{3}\frac{9!}{2!\,2!} + \binom{5}{4}\frac{8!}{2!} - 7!\,.$$