Combinations with Repetition

Daniel H. Luecking

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- Case3. The prizes are identical and no contestant can receive more than 1. This is a combination: $C(10, 4) = 10!\,/(4!\,6!\,)$ possible outcomes.
- Case 4. The prizes are identical and a contestant can receive any or all of them. We'll see that there are $C(13, 4)$ possible outcomes, but it is not so obvious why.

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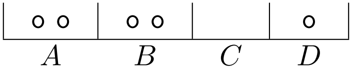
In the following picture we imagine a box for each contestant and a marble for each prize. A marble in a box means a prize for that contestant:



This means we can encode an outcome by a string of o's and $|$'s. Since the sequence always starts and ends with a $|$ we can omit these and the above outcome looks like oo$|$oo$||$o. The case where A gets all 5 prizes would correspond to ooooo$|||$.

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$$\frac{(k+n-1)!}{k!\,(n-1)!} = C(k+n-1,k).$$

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Returning to our original problem (4 identical prizes divided among 10 contestants, with repetition): Case 4 can be done in $C(4+10-1,4) = 13!\,/(4!\,9!\,)$ ways.

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In general if an equation has has n variables and their sum equals k, then the number of (nonnegative integer) solutions is $C(k+n-1, k)$.

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Suppose I want each employee to get at least \$100. If we want, say $b_1 \geq 1$, then $b_1 = q_1 + 1$, where $q_1 \geq 0$. So the equation for the b's becomes the following equation for the q's:

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Alternatively, just give everyone \$100 and then select 7 times, with repetition from a set of size 8.

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A voting system where you rate each candidate with 'disapprove', 'approve' or 'no opinion' would yield 3^{10} possible outcomes.

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Thus there are $26! - 24! - 24! + 22!$ permutations containing neither.