Permutations and Combinations

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So the number of arrangements is

$$\frac{10!}{3!\,7!} \cdot \frac{7!}{2!\,5!} \cdot \frac{5!}{2!\,3!} \cdot 3 \cdot 2 \cdot 1 = \frac{10!}{3!\,2!\,2!}$$

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`_V_E_O_B_A_Y_`

where the "_" indicate places the `"L"`s might be inserted. Since we don't want any consecutive `"L"`s we have to choose 4 of those 7 spaces and put an `"L"` in each.

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Thus:
task 1: $6!$ ways,
task 2: $C(7,4)$ ways.
Rule of product: $6!\,C(7,4) = 6!\,\dfrac{7!}{4!\,(7-4)!} = 25{,}200$

Miscellaneous

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The binomial theorem

Combinations come up in an unexpected way in algebra: the formula for $(x+y)^n$:

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k \,.$$

There is a way to justify this formula using combinations. When we multiply out

$$(x+y)(x+y)(x+y)(x+y)\cdots(x+y)$$

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For example:

$$\begin{aligned}(x+y)^3 &= [(x+y)(x+y)](x+y)\\ &= [x(x+y)+y(x+y)](x+y) = [xx+xy+yx+yy](x+y)\\ &= xx(x+y)+xy(x+y)+yx(x+y)+yy(x+y)\\ &= xxx+xxy+xyx+xyy+yxx+yxy+yyx+yyy\\ &= x^3+3x^2y+3xy^2+y^3\end{aligned}$$