# Basic Principals of Combinatorics

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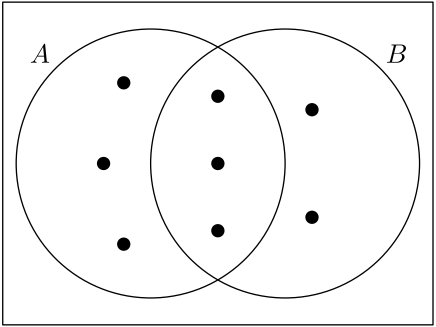
   - If any one task is done differently, the final outcome is different.
   - All final outcomes are produced by some way of performing the sequence of tasks.

## A picture of one-to-one correspondence

$$\begin{array}{ccc}
1 & \longleftrightarrow & A \\
2 & \longleftrightarrow & B \\
3 & \longleftrightarrow & C \\
4 & \longleftrightarrow & D \\
\vdots & \vdots & \vdots \\
24 & \longleftrightarrow & X \\
25 & \longleftrightarrow & Y \\
26 & \longleftrightarrow & Z
\end{array}$$

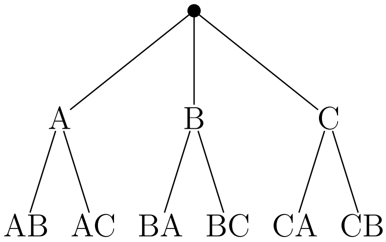
## A picture of the rule of sum




$|A| = 6$, $|B| = 5$, $|A \cap B| = 3$, $|A \cup B| = 6 + 5 - 3 = 8$.

## A picture of the rule of product

The following illustrates building strings of length $2$ from the letters A, B, and C without repeated letters. This illustrates the rule of product: the first task (select a letter) produces a 3-way branching, and then the second task (select a different letter) produces 2-way branching. There are a total of $3 \cdot 2$ paths to the different final results.



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If the set we are choosing from is an $n$-set and we are choosing $k$ elements, we refer to these as *permutations of $n$ things taken $k$ at a time*. (The shorter '*$k$-permutations of an $n$-set*' is also used.)

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The Rule of product comes to our aid here: we can build a $k$-permutation in $k$ steps: Pick the element for each position one-by-one.

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The special case $P(n,n)$ is important: $P(n,n) = n(n-1)(n-2)\cdots 1$. We have a special notation for this product, called a *factorial*:

$$n! = n(n-1)(n-2)\cdots 1 \quad \text{(special case: } 0! = 1\text{)}$$

With this notation we have the shorter formulas:

$$P(n,n) = n! \quad \text{and} \quad P(n,k) = \frac{n!}{(n-k)!}.$$

## Other ways to look at permutations

Note that $P(100, 3) = 100!\,/97!\, = 100 \cdot 99 \cdot 98 = 970200$. The first computation (dividing $100!$ by $97!$) is impossible to do by hand, and not even guaranteed to be exact in some computer programs, the second is easy and pretty quick.

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Consider the set $S = \{A, B, C, D, E, F, G\}$ and the following table:

| $x$ | 1 | 2 | 3 | 4 |
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In fact, there is a one-to-one correspondence between permutations and one-to-one functions. That is, for any $k$-set $A$ and $n$-set $S$, $P(n, k)$ is the number of one-to-one functions from $A$ to $S$.

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If we group all the permutations into clusters of size $6$ that each represent the same subset, we get $P(10, 3)/P(3, 3) = \frac{10 \cdot 9 \cdot 8}{3 \cdot 2 \cdot 1} = 120$ subsets.

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Since there are $2!$ ways to permute the `O`s, $2!$ ways to permute the `K`s and $3!$ ways to permute the `E`s, we have $2!\,2!\,3!$ different permutations (of the tiles) that produce the same string.

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How many ways can we arrange the letters of the string `"BOOKKEEPER"`?

To be clear what this means, imagine we write each letter on a tile, `B` `O` `O` `K` `K` `E` `E` `P` `E` `R`, then we shuffle these tiles and line them up, *recording the resulting string* of letters. How many different strings do we get?

If we count each tile as different, we get $10!$ permutations. But if, for example, all we do is exchange the two `O`-tiles we don't get different strings. Same for the `K`-tiles and `E`-tiles.

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## Other examples

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What about "BEE"? $\frac{3!}{2!} = 3$ and they are "BEE", "EBE" and "EEB".

Finally, what about "STRATIFICATIONAL" and "GASTROENTEROLOGIST"? Give it a try, the hard part is not missing any repetitions.

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A similar argument applies to the following kind of problem: how many permutations of the 26 letters of the alphabet contain the substrings `"CAT"`, `"DOG"` and `"LYNX"`? Answer: $19!$, the number of permutations of `C A T`, `D O G`, `L Y N X`, and 16 individual letters.

How about permutations of the 26 letters of the alphabet contain the substrings `"CAT"`, `"DOG"` and `"KID"`? Answer: $20!$, the number of permutations of `C A T`, `K I D O G`, and 18 individual letters.

## Selections without order

A *combination of $n$ things taken $k$ at a time* is subset of size $k$ from an $n$-set. The distinction between a permutation and a combination is that 2 combinations are the same if they have the same elements, while 2 permutations are the same if they have the same elements in the same order.

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More generally, if we view a permutation as a one-to-one function, from some $k$-set $A$ into an $n$-set $S$ then, to get a permutation, we have to choose the correct number of elements of $S$ ($k$ of them) and then associate each of those with an element of $A$.

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**Definition**

$C(n,k)$ stands for the number of combinations possible when choosing $k$ elements from a set of size $n$.

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Example: I have 7 books I haven't read and I want to take 3 of them on vacation. There are

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$$C(7,2)C(5,2) = \frac{7!}{5!\ 2!}\,\frac{5!}{3!\,2!} = \frac{7 \cdot 6}{2 \cdot 1}\,\frac{5 \cdot 4}{2 \cdot 1} = 210 \text{ ways to do this.}$$